MATRICES

What is it?

- ✓ Matrix algebra is a means of making calculations upon arrays of numbers (or data).
- ✓ Most data sets are matrix-type

Why use it?

- ✓ Matrix algebra makes *mathematical expression and computation* easier.

1.1 Matrices

Consider the following set of equations:

$$\begin{cases} x+y=7, \\ 3x-y=5. \end{cases}$$

It is easy to show that $x = 3$ and $y = 4$.

HOW ABOUT SOLVING

$$\begin{cases} x+y-2z=7, \\ 2x-y-4z=2, \\ -5x+4y+10z=1, \\ 3x-y-6z=5. \end{cases}$$

Matrices can help…

Basic concept of matrix

- ✓ Here, x, y, z are unknowns and their coefficients are all numbers. Arranging the coefficients in which they occur in the equation and enclosing them in square brackets. We obtain a rectangular array of the form

$$\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & -4 \\ -5 & 4 & 10 \\ 3 & -1 & 6 \end{bmatrix}$$

- ✓ This rectangular array is an example of a matrix. The horizontal lines are called **rows** and vertical lines are called **columns**. There are 4 rows and 3 columns in the matrix. Therefore it is a matrix of **order 4 x 3**. The numbers 1,2, -2,…etc. in this matrix are called its **elements**.

Basic concept of matrix

- A matrix is not a number. It has got no numerical value. It is just an ordered collection of numbers arranged in the form of a rectangular array.

Example: Simply 5 is a number. But in our notation of matrix [5] is a matrix of the order 1 x 1 and we cannot have 5=[5]. We cannot have a relation of equality between a matrix and a number.

DEFINITIONS - MATRIX

A system of *m n* numbers (real or complex) arranged in the form of a rectangular array having *m* rows and *n* columns is called an *m* x *n* matrix. (read as *m* by *n* matrix) .

An *m* x *n* matrix is usually written as

$$\begin{bmatrix} a_{11} & a_{12} & \cdots\cdots\cdots & a_{1n} \\ a_{21} & a_{22} & \cdots\cdots\cdots & a_{2n} \\ & & \vdots & \\ & & \vdots & \\ a_{m1} & a_{m2} & \cdots\cdots\cdots & a_{mn} \end{bmatrix}_{mxn}$$

Ex- $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & 1 & 2 \end{bmatrix}$ is a matrix of the type (order) 2 x 4.

Matrices

In the matrix numbers a_{ij} are called *elements*. First subscript indicates the row; second subscript indicates the column. The matrix consists of mn elements

Ex- what is the type of the matrix given below: $A=\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & 1 & 2 \\ 4 & 8 & 5 & 6 \\ 2 & 7 & 6 & 7 \end{bmatrix}$?

Write the elements a_{11} , a_{24} and a_{31} for this matrix.

Types of Matrices

- Row Matrix: An m x n matrix is called row matrix if m = 1. Ex: A = $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \end{bmatrix}$
- Column Matrix: An m x n matrix is called row matrix if n = 1. Ex: A = $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$
- Square Matrix: A square matrix is a matrix that has the same number of rows and columns i.e. if $m = n$.

  Ex: A= $\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$ is a square matrix of order 2 x 2.

Types of Matrices

- Zero Matrix: A matrix each of whose elements is zero & is called a zero matrix. It is usually denoted by "O". It is also called *"Null Matrix"*

$$A = \begin{bmatrix} 0 & 0 & & 0 \\ 0 & 0 & & 0 \\ & & & \\ 0 & 0 & & 0 \end{bmatrix}$$

Types of Matrices

- **Diagonal Matrix:** A square matrix with its all non *diagonal elements* as zero. i.e if $A = [a_{ij}]$ is a diagonal matrix, then $a_{ij} = 0$ whenever $i \neq j$. ***Diagonal elements*** are the a_{ij} elements of the square matrix A for which $i = j$.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Types of Matrices

- Diagonal elements are said to constitute the **main diagonal** or **principal diagonal** or simply a **diagonal**.
- The diagonals which lie on a line perpendicular to the diagonal are said to constitute **secondary diagonal.**

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Here main diagonal consists of 1 & 4 and secondary diagonal consists of 2 & 3.

Types of Matrices

- Scalar Matrix: It's a diagonal matrix whose all elements are equal.

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Question: (T/F) 1. Every scaler matrix is diagonal matrix. 2. Every diagonal matrix is a scaler matrix.

- Unit Matrix: It's a scalar matrix whose all diagonal elements are equal to unity. It is also called a Unit Matrix or Identity Matrix. It is denoted by I_n.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Types of Matrices

- Triangular Matrix: If every element above or below the diagonal is zero, the matrix is said to be a triangular matrix.

$$\begin{bmatrix} 1 & 4 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$ Upper Triangular Matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 4 & 0 \\ 3 & 2 & 1 \end{bmatrix}$$ Lower Triangular Matrix

Equality of Matrices

- Two matrices A & B are said to be equal iff:
  1. A and B are of the same order
  2. All the elements of A are equal as that of corresponding elements of B
- Two matrices A = [aij] & B = [bij] of the same order are said to be equal if aij = bij

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, B = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$$

If A & B are equal, then

x=1, y=2, z=3, w=4

Equality of Matrices (Problems for Practice)

Qu 1. If $\begin{bmatrix} x+3 & 3y+x \\ z-1 & 4a-6 \end{bmatrix} = \begin{bmatrix} 0 & -7 \\ 3 & 2a \end{bmatrix}$ Find the values of x, y, z, a.

Qu 2. If $\begin{bmatrix} x & 6y \\ z-1 & 2a \end{bmatrix} = \begin{bmatrix} 4 & 9 \\ 8 & 2 \end{bmatrix}$ Find the values of x, y, z, a.

Trace of a Matrix

- In a square matrix A, the sum of all the diagonal elements is called the trace of A. It is denoted by tr A. $i.e.\ tr\ A = \sum_{i=1}^{n} a_{ii}$

Ex- $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 4 & 0 \\ 3 & 2 & 1 \end{bmatrix}$ then

tr A= 1+4+1 =6

Operations on Matrices

- Addition/Subtraction
- Scalar Multiplication
- Matrix Multiplication

Addition and Subtraction

- Two matrices may be added (or subtracted) iff they are the same order.

- Simply add (or subtract) the corresponding elements. So, **A + B = C**

Addition and Subtraction (cont.)

- Where

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ c_{31} & c_{32} \end{bmatrix}$$

$$a_{11} + b_{11} = c_{11}$$
$$a_{12} + b_{12} = c_{12}$$
$$a_{21} + b_{21} = c_{21}$$
$$a_{22} + b_{22} = c_{22}$$
$$a_{31} + b_{31} = c_{31}$$
$$a_{32} + b_{32} = c_{32}$$

Addition / Subtraction (Problems for Practice)

Q: If A = $\begin{bmatrix} 3 & 8 & 11 \\ 6 & -3 & 8 \end{bmatrix}$ and B = $\begin{bmatrix} 1 & -6 & 15 \\ 3 & 8 & 17 \end{bmatrix}$

find A+B, A-B.

Scalar Multiplication

- To multiply a scalar times a matrix, simply multiply each element of the matrix by the scalar quantity

$$k\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} ka_{11} & ka_{12} \\ ka_{21} & ka_{22} \end{bmatrix}$$

- Ex: If A $\begin{bmatrix} 3 & 8 & 11 \\ 6 & -3 & 8 \end{bmatrix}$, then
=

$$10\text{A} = \begin{bmatrix} 30 & 80 & 110 \\ 60 & -30 & 80 \end{bmatrix}$$

Problems for Practice

Q1: If $A = \begin{bmatrix} 1 & 3 & 4 \\ -2 & 4 & 8 \\ 3 & -2 & -1 \end{bmatrix}$ and $B \begin{bmatrix} 4 & 1 & 0 \\ 1 & 3 & 5 \\ 0 & 1 & 6 \end{bmatrix}$

find 5A+2B.

Q2: If $A = \begin{bmatrix} 3 & 8 & 11 \\ 6 & -3 & 8 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -6 & 15 \\ 3 & 8 & 17 \end{bmatrix}$

find 7A - 5B.

Q3: If $A = \begin{bmatrix} 2 & -2 & 7 \\ 4 & 6 & 3 \end{bmatrix}$; find matrix X such that

X+A=O where O is a null matrix.

Problems for Practice

Q4: If $A = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 7 \\ 5 & 3 \end{bmatrix}$

Show that 5(A+B) = 5A + 5B.

Q5: If $A = \begin{bmatrix} 1 & 2 & 0 & 4 \\ 2 & 4 & -1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 & 0 & 3 \\ 1 & -2 & 2 & 3 \end{bmatrix}$

find a 2 x 4 matrix "X" such that A - 2X = 3B.

Q6: If $X+Y = \begin{bmatrix} 7 & 0 \\ 2 & 5 \end{bmatrix}$ and $X - Y = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$ Find X&Y.

Matrix Multiplication

- If $A = [a_{ij}]$ is a $m \times p$ matrix and $B = [b_{ij}]$ is a $p \times n$ matrix, then AB is defined as a $m \times n$ matrix $C = AB$, where $C= [c_{ij}]$ with

$$c_{ij} = \sum_{k=1}^{p} a_{ik} b_{kj} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{ip}b_{pj}$$ for $1 \le i \le m,\ 1 \le j \le n$.

Example: $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 2 \\ 2 & 3 \\ 5 & 0 \end{bmatrix}$ and $C = AB$.

Evaluate c_{21}.

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 2 & 3 \\ 5 & 0 \end{bmatrix}$$

$$c_{21} = 0 \times (-1) + 1 \times 2 + 4 \times 5 = 22$$

Rule of Matrix Multiplication

- Multiplication or Product of two matrices A & B is possible iff the number of columns of A is equal to the number of rows of B.
- The rule of the multiplication of the matrices is row-column wise (→↓).
- The first row of AB is obtained by multiplying the 1^{st} row of A with 1^{st}, 2^{nd} & 3^{rd} column of B.
- The second row of AB is obtained by multiplying the 2^{nd} row of A with 1^{st}, 2^{nd} & 3^{rd} column of B.
- The third row of AB is obtained by multiplying the 3^{rd} row of A with 1^{st}, 2^{nd} & 3^{rd} column of B.

Matrix Multiplication

Example: $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 2 \\ 2 & 3 \\ 5 & 0 \end{bmatrix}$, Evaluate $C = AB$.

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 2 & 3 \\ 5 & 0 \end{bmatrix} \Rightarrow \begin{cases} c_{11} = 1\times(-1) + 2\times 2 + 3\times 5 = 18 \\ c_{12} = 1\times 2 + 2\times 3 + 3\times 0 = 8 \\ c_{21} = 0\times(-1) + 1\times 2 + 4\times 5 = 22 \\ c_{22} = 0\times 2 + 1\times 3 + 4\times 0 = 3 \end{cases}$$

$$C = AB = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 2 & 3 \\ 5 & 0 \end{bmatrix} = \begin{bmatrix} 18 & 8 \\ 22 & 3 \end{bmatrix}$$

PROBLEMS FOR PRACTICE

Q1: If $A = \begin{bmatrix} 2 & 3 & 2 \\ 4 & 3 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 9 \\ 0 & 1 \\ 6 & 9 \end{bmatrix}$. Find AB.

Q2: If $A = \begin{pmatrix} 1 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & -2 & -1 \\ 6 & 12 & 6 \\ 5 & 10 & 5 \end{pmatrix}$

Show that AB is a null matrix & BA is not a null matrix.

Problems for Practice

Q3: If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 1 \\ 4 & 5 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$

Show that A(BC) = (AB)C

Problems for Practice

Q5: Find “x” such that :

$$\begin{bmatrix} 1 & x & 1 \end{bmatrix} \begin{pmatrix} 1 & 3 & 2 \\ 2 & 5 & 1 \\ 15 & 3 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ x \end{pmatrix} = O$$

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Problems for Practice

Problems for Practice

Q6: If $A = \begin{pmatrix} 1 & 3 & 2 \\ 0 & 5 & 7 \\ 6 & 4 & 8 \end{pmatrix}$ Find $A^2 + 7A + 3I$

Q7: If $A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$ Prove that $A^2 = 4A + 5I$

Properties of Matrices

Matrices A, B and C are square matrix of same order,

$A + B = B + A$ (commutative law)

$A + (B + C) = (A + B) + C$ (associative law)

$\lambda(A + B) = \lambda A + \lambda B$, where λ is a scalar (distributive law)

Properties of Matrices

Matrices A, B and C are square matrix,

$A(B + C) = AB + AC$

$(A + B)C = AC + BC$

$A(BC) = (AB)\ C$

However

$AB \neq BA$ in general

$AB = 0$ NOT necessarily imply $A = 0$ or $B = 0$

$AB = AC$ NOT necessarily imply $B = C$

Transpose of a Matrix

The matrix obtained by interchanging the rows and columns of a matrix A is called the transpose of A (written as A^T or $A`$).

Example: $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$

The transpose of A is $A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$

For a matrix $A = [a_{ij}]$, its transpose $A^T = [b_{ij}]$, where $b_{ij} = a_{ji}$.

Practice Problems

Q1: If $A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$ $B = \begin{bmatrix} 3 & 1 \\ 2 & 5 \end{bmatrix}$

Find $A' + B'$, $(A+B)'$, $A'B'$

Q2: Verify that $(AB)' = B'A'$ if

If $A = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 2 & -3 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 0 \\ 1 & -1 \\ 0 & -2 \end{bmatrix}$

Symmetric & Skew Symmetric Matrices

A matrix A such that $A^T = A$ is called symmetric, i.e., $a_{ji} = a_{ij}$ for all i and j.

- ✓ $A + A^T$ must be symmetric. Why?

Ex- $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 6 \\ 3 & 6 & 9 \end{bmatrix}$ is symmetric.

A matrix A such that $A^T = -A$ is called skew-symmetric, i.e., $a_{ji} = -a_{ij}$ for all i and j.

- ✓ $A - A^T$ must be skew-symmetric. Why?
- ✓ **The principal diagonal elements of skew- symmetric matrix are zero.**

Practice Problems

Q1: Express $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ as a sum of symmetric & skew symmetric matrix.

Q2: If $A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$, then prove that

i) $A+A'$ is a symmetric matrix

ii) $A - A'$ is a skew symmetric matrix

iii) AA' & $A'A$ are symmetric matrices

Q3: Express $\begin{bmatrix} 6 & 1 \\ 3 & 4 \end{bmatrix}$ as a sum of symmetric & skew symmetric matrix.

Idempotent Matrix:

A square matrix A is said to be idempotent matrix if $A^2 = A$

Ex: $A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & 3 \end{bmatrix}$ is an idempotent matrix.

Involutory Matrix:

A square matrix A is said to be involutory matrix if $A^2 = I$ where I is an identity matrix.

Ex: $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ is an involutory matrix.

Nilpotent Matrix:

A square matrix A is said to be nilpotent matrix of index m if it satisfy the relation

$A^m = O$ and $A^{m-1} \neq O$

where m is positive integer and O is null matrix.

Ex: $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ are nilpotent matrices.

1.5 Determinants

Determinant of order 2

Consider a 2×2 matrix: $$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Determinant of A, denoted $|A|$ is a number and can be evaluated by

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

1.5 Determinants

Determinant of order 2

easy to remember (for order 2 only)..

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = {}^{+}a_{11}a_{22} - a_{12}a_{21}$$

- +

Example: Evaluate the determinant: $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$

$$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 1 \times 4 - 2 \times 3 = -2$$

Practice Problems

Q1: Find the determinant of :

(i)

$$\begin{vmatrix} 6 & 8 \\ 7 & -2 \end{vmatrix}$$

(ii)

$$\begin{vmatrix} 4 & 0 \\ 1 & 0 \end{vmatrix}$$

1.5 Determinants of order 3

Consider an example: $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

Its determinant can be obtained by:

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 3\begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} - 6\begin{vmatrix} 1 & 2 \\ 7 & 8 \end{vmatrix} + 9\begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix}$$

$$= 3(-3) - 6(-6) + 9(-3) = 0$$

You are encouraged to find the determinant by using other rows or columns

Practice Problems

Find the value of

i) $\begin{vmatrix} 3 & -5 & 4 \\ 7 & 6 & 1 \\ 1 & 2 & 3 \end{vmatrix}$

ii) $\begin{vmatrix} 1 & 4 & 7 \\ -2 & 3 & 4 \\ 1 & 4 & -4 \end{vmatrix}$

1.5 Determinants

The following properties are true for determinants of *any* order.

1. If every element of a row (column) is zero, e.g., $\begin{vmatrix} 1 & 2 \\ 0 & 0 \end{vmatrix} = 1\times 0 - 2\times 0 = 0$, then $|A| = 0$.

2. $|A^T| = |A|$ ← determinant of a matrix = that of its transpose

3. $|AB| = |A||B|$

4. $|kA| = k^n|A|$

Singular Matrix

A square matrix A is said to be singular if its determinant is zero. $i.e.$ $|A| = 0$

- It is said to be non- singular matrix if $|A| \neq 0$.

Example: $A = \begin{bmatrix} 5 & 3 \\ 10 & 6 \end{bmatrix}$, Here $|A|=0$

So, Matrix A is singular.

Theorem: Show that every skew symmetric matrix of an odd order is singular.

Proof: Let *A* be a skew symmetric matrix of an odd order *n* where *n* is odd then $A' = -A$

$$|A'| = |-A|$$

$$|A'| = (-1)^n |A| \quad \{ |kA| = k^n |A| \}$$

$$|A| = -|A| \quad \{ |A'| = |A| \}$$

$$2|A| = O$$

$$|A| = O$$

Hence, A is singular.

Question: If A is square matrix of order 4 such that $|A| = 3$ then find the value of $|2A|$.

MATRIX

Minor

The minor of an element a_{ij} of a determinant of order *n* x *n* is the determinant of order *(n-1)* x *(n-1)* obtained by leaving i^th^ row and j^th^ column. It is denoted M_{ij}.

<u>Example:</u> $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

M_{21} = The minor of a_{21}= $\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} = a_{12}a_{33} - a_{32}a_{13}$

Similarly, also find M_{23}, M_{31}.

Cofactors

The cofactor of an element a_{ij} of a determinant is equal to $(-1)^{i+j}$ times of its minor. The cofactor a_{ij} is denoted A_{ij}.

$$i.e.\ A_{ij} = (-1)^{i+j} M_{ij}$$

Example: $A = \begin{vmatrix} 3 & 7 & 8 \\ -2 & 5 & 0 \\ 1 & 4 & 2 \end{vmatrix}$

A_{32} = The cofactor of 4 = $(-1)^{3+2} M_{32} = (-1)^{3+2} \begin{vmatrix} 3 & 8 \\ -2 & 0 \end{vmatrix} = -16$

Find A_{13} and A_{22}.

Note: 1. The sum of products of elements of a row (or column) with their co-factors is equal to the value of the determinant.

$$\Delta = a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in} \ (i = 1,2, \dots, n)$$

Or

$$\Delta = a_{1j}A_{1j} + a_{2j}A_{2j} + \cdots + a_{nj}A_{nj} \ (j = 1,2, \dots, n)$$

2. In a determinant ,the sum of products of elements of a row (or column) with the co-factors of any other row (column) is zero.

Example: $\Delta = \begin{vmatrix} 2 & -3 & 5 \\ 3 & 7 & 0 \\ 4 & 1 & 8 \end{vmatrix}$

Calculate

$$\Delta = a_{12}A_{13} + a_{22}A_{23} + a_{32}A_{33} = ?$$

$$\Delta = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} = ?$$

Adjoint of a square matrix

Let *A* be *n* x *n* matrix. The transpose B' of the matrix $B = [A_{ij}]_{nxn}$

Where A_{ij} denotes the co-factors of the element a_{ij} in the determinant of the matrix *A* and is denoted by the symbol *Adj A*.

i.e.

$$If \quad A = \begin{bmatrix} a_{11} & a_{12} \dots \dots \dots a_{1n} \\ a_{21} & a_{22} \dots \dots \dots a_{2n} \\ & \vdots \\ & \vdots \\ a_{n1} & a_{n2} \dots \dots \dots a_{nn} \end{bmatrix}_{nxn}$$

Then *Adj A*= The transpose of the matrix $\begin{bmatrix} A_{11} & A_{12} \cdots\cdots\cdots & A_{1n} \\ A_{21} & A_{22} \cdots\cdots\cdots & A_{2n} \\ & \vdots & \\ A_{n1} & A_{n2} \cdots\cdots\cdots & A_{nn} \end{bmatrix}$

$$\text{Adj } A == \begin{bmatrix} A_{11} & a_{21} \cdots\cdots\cdots & a_{n1} \\ A_{12} & a_{22} \cdots\cdots\cdots & a_{n2} \\ & \vdots & \\ a_{1n} & a_{2n} \cdots\cdots\cdots & a_{nn} \end{bmatrix}$$

Example: If $A = \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & -1 \\ 2 & 0 & 4 \end{bmatrix}$ then find *Adj A*

Find firstly cofactors and generate $B = \begin{bmatrix} 4 & -2 & -2 \\ -4 & -2 & 2 \\ -4 & 1 & 1 \end{bmatrix}$

Then $Adj\ A = B' = \begin{bmatrix} 4 & -4 & -4 \\ -2 & -2 & 1 \\ -2 & -2 & 1 \end{bmatrix}$

Question: If A $= \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then find *Adj A*.

Theorem: If $A = [a_{ij}]$ be square matrix of order n then

$$A.Adj\ A = Adj\ A.A = |A|I_n$$

Question: If A $= \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then find Adj(*Adj A)*.

Question: Find *Adj A* where $A = \begin{bmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & -6 & 7 \end{bmatrix}$ and verify

$$A.Adj\ A = Adj\ A.A = |A|I_3$$

Inverse of a Matrix

Let *A* be a non-singular square matrix of order *n*. If there exist a square matrix of order *n* such that $A.\ B = B.\ A = I$

Where *I* is identity matrix of order *n*.

Then *B* is called as the inverse of *A*. The inverse of a matrix *A* is denoted by A^{-1}.

Since, we know that $A.Adj\ A = Adj\ A.A = |A|I_n$

$$\text{then } \frac{A.Adj\ A}{|A|} = I \quad if \;\; |A| \neq 0$$

Hence, $$A^{-1} = \frac{Adj\ A}{|A|} = I \quad if \;\; |A| \neq 0$$

Example 1. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Question 1. If $A.\begin{bmatrix} -2 & 1 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} -1 & 7 \\ -1 & 20 \end{bmatrix}$ Find the matrix A.

Question 2. Find the inverse of the matrix $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$

Elementary Transformation (Operations)

An elementary transformation is an operation of any one of the following types:

Type-I: The interchange of i^{th} and j^{th} rows (or columns) of the matrix.

This operation is denoted by $R_i \leftrightarrow R_j$ or $C_i \leftrightarrow C_j$ or by $R_{ij}(C_{ij})$.

Let $A = \begin{bmatrix} 3 & 1 & 9 \\ 7 & 5 & 4 \\ 9 & 0 & 8 \end{bmatrix}$ by interchanging first and third rows of A, the matrix is obtained $\begin{bmatrix} 9 & 0 & 8 \\ 7 & 5 & 4 \\ 3 & 1 & 9 \end{bmatrix}$. This operation is denoted by $R_1 \leftrightarrow R_3$

Type-II: The i^{th} row (or column) of the matrix is multiplied by scalar $k \neq 0$ then this row (or column) operation is denoted by $R_i \rightarrow kR_i$ or $C_i \rightarrow kC_i$.

Multiplying the second row of the above matrix A by 2 then the matrix is obtained $\begin{bmatrix} 3 & 1 & 9 \\ 14 & 10 & 8 \\ 9 & 0 & 8 \end{bmatrix}$. This row operation is denoted by $R_2 \to 2R_2$.

Type-III: The $k \neq 0$ times of the elements of the i^{th} row (or column) and added to the corresponding elements of the j^{th} row (or column). Then this row (or column) operation is denoted by $R_j \to R_j + kR_i$ or

$C_j \to C_j + kC_i$.

In above matrix, adding three times the elements of the first row to the corresponding elements of third row in above matrix A. The matrix A is obtained $\begin{bmatrix} 3 & 1 & 9 \\ 7 & 5 & 4 \\ 18 & 3 & 35 \end{bmatrix}$. This row operation is denoted by

$R_3 \to R_3 + 3R_1$.

Equivalent Matrix

Two matrices A and B of same order are said to be equivalent if one is obtained from other by elementary operation. In this case we write $A \sim B$.

Ex: $A = \begin{bmatrix} 3 & 1 & 9 \\ 7 & 5 & 4 \\ 9 & 0 & 8 \end{bmatrix}, B = \begin{bmatrix} 9 & 0 & 8 \\ 7 & 5 & 4 \\ 3 & 1 & 9 \end{bmatrix}$ then A ~ B.

Elementary Matrix

A matrix obtained from a unit matrix by a single elementary operation is called elementary matrix.

Ex: $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ applying $R_2 \to 4R_2$ then matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is an elementary matrix.

Inverse of a matrix from elementary operations (Gauss- Jordan Method)

Let A be non-singular matrix. We know that $A = I \, . \, A$

Applying a sequence of row transformation in such a way A is changed to I and I is changed to B .

i.e. $I = B \, . \, A$

Multiply by A^{-1} both sides

$$IA^{-1} = BAA^{-1}$$

$$A^{-1} = BI$$

$$A^{-1} = B$$

B is the inverse of A.

Example 1: Find the inverse of the matrix by using row elementary transformation method $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$

Solution: Since $A = I \,.\, A$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{A}$$

Applying $R_2 \to R_2 - 2R_1, \;\; R_3 \to R_3 - 3R_1$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & -1 \\ 0 & -1 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \text{A}$$

Applying $R_2 \leftrightarrow R_3$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -3 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 0 & 1 \\ -2 & 1 & 0 \end{bmatrix} \text{A}$$

Applying $R_1 \to R_1 + 3R_3, \quad R_2 \to R_2 - 3R_3$

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} -5 & 3 & 0 \\ 3 & -3 & 1 \\ -2 & 1 & 0 \end{bmatrix} \text{A}$$

Applying $R_1 \to R_1 + 2R_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 2 \\ 3 & -3 & 1 \\ -2 & 1 & 0 \end{bmatrix} \text{A}$$

Applying $R_2 \to (-1)R_2, \quad R_3 \to (-1)R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{bmatrix} \text{A}$$

$$I = B\,A$$

$$A^{-1} = \begin{bmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{bmatrix} = B$$

Example 2: Find the inverse of the matrix by using row elementary transformation method $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$

Solution: Since $A = I \cdot A$

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Applying $R_1 \leftrightarrow R_2$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Applying $R_3 \rightarrow R_3 - 3R_1$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & -5 & -8 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -3 & 1 \end{bmatrix} A$$

Applying $R_3 \to R_3 + 5R_2$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 5 & -3 & 1 \end{bmatrix} A$$

Applying $R_1 \to R_1 - 2R_2$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 0 \\ 5 & -3 & 1 \end{bmatrix} A$$

Applying $R_3 \to \frac{1}{2} R_3$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 0 \\ \frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \end{bmatrix} A$$

Applying $R_2 \to R_2 - 2R_3$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ -4 & 3 & -1 \\ \frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \end{bmatrix} A$$

Applying $R_1 \rightarrow R_1 + R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -4 & 3 & -1 \\ \frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \end{bmatrix} A$$

$$I = B\,A$$

$$A^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -4 & 3 & -1 \\ \frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \end{bmatrix} = B$$

Question: Find inverse of matrix by row elementary transform method

$(i)\begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$ $(ii)\begin{bmatrix} 1 & -1 & 1 \\ 4 & 1 & 0 \\ 8 & 1 & 1 \end{bmatrix}$

RANK OF A MATRIX

Sub matrix of a Matrix

Suppose *A* is any matrix of type *m* x *n*, then a matrix obtained by leaving some rows and columns from *A* is called a submatrix of *A*.

Example: $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 3 & 5 & 2 & 9 \end{bmatrix}_{3\times 4}$

Then submatrix of *A* are $\begin{bmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ 3 & 5 & 2 \end{bmatrix}_{3\times 3}, \begin{bmatrix} 3 & 4 \\ 7 & 8 \\ 2 & 9 \end{bmatrix}_{3\times 2}$ ….. etc.

Note: The matrix A itself is a submatrix of A because it is obtained from a by leaving no rows and no columns.

Minor of a Matrix

Let *A* be an *m* x *n* matrix then the determinant of every square sub-matrix of *A* is called a minor of the matrix *A*.

Example: $A = \begin{bmatrix} 3 & 5 & 7 & 8 \\ 2 & 2 & 5 & 3 \end{bmatrix}_{2\times 4}$

Then $\begin{vmatrix} 3 & 5 \\ 2 & 2 \end{vmatrix}$, $\begin{vmatrix} 5 & 7 \\ 2 & 5 \end{vmatrix}$, $\begin{vmatrix} 7 & 8 \\ 5 & 3 \end{vmatrix}$ .... etc. are minors of order 2 of *A*.

Note: If *A* is *m* x *n* matrix then

A has minors of order $\leq$ minimum of *(m, n)*.

- In previous example,

The highest order of the minor of a matrix *A* is 2.

Rank of a Matrix

Let A be m x n matrix then a natural number (including 0) r is said to be rank of a matrix A if it satisfies the following two properties:

(i) There is at least one square sub-matrix of A of order r whose determinant is not equal to zero.

(*i.e.* There is at least one minor of A of order r which is non-vanishing.)

(ii) If the matrix A contains any square submatrix of order $r + 1$, then the determinant of every square submatrix of A of order $r + 1$ should be zero. (*i.e.* Every minor of A of order $r + 1$ and higher order vanishes.)

We shall denote the rank of a matrix A by the symbol $\rho(A)$.

$i.e.$ $\rho(A)$ =The rank of a matrix is the order of any highest order non- vanishing minor of the matrix.

Note: 1. If A is m x n matrix. Then rank of A $\leq$ minimum of (m, n).

2. If A is n x n non-singular matrix i.e. $|A| \neq 0$

then $\rho(A) = n$.

Ex- Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ we know that |A| =1

Then $\rho(A) = 3$

3. Zero (0) is the only rank of zero matrix

$i.e.$ If A is zero matrix then $\rho(A) = 0$

Ex- Let $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ then $\rho(A) = 0$.

Example 1. Find the rank of $A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 0 & 2 \end{bmatrix}_{2\times 3}$

<u>Solution:</u> Since, A is of order 2 x 3.

Then rank of A $\leq$ min *(2, 3)*

rank A $\leq$ 2

Consider, minor of order 2

$$\begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix} = -1 \neq 0$$

Hence, $\rho(A) = 2.$

Example 2. Find the rank of $A = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 0 & 2 \\ 3 & -1 & 3 \end{bmatrix}_{3\times 3}$

Solution: Since, A is of order 3 x 3.

Then rank of A $\leq$ min *(3, 3)*

rank A $\leq$ 3

Consider, minor of order 3

$$|A| = \begin{vmatrix} 2 & -1 & 1 \\ 1 & 0 & 2 \\ 3 & -1 & 3 \end{vmatrix}$$

$$= 2(0+2) - (-1)(3-6) + 1(-1+0) = 4 - 3 - 1 = 0$$

So, $\rho(A) \neq 3$ (Since, all 3rd order minor vanishes)

Again, consider minor of order 2

$\begin{vmatrix} 2 & -1 \\ 1 & 0 \end{vmatrix} = 1 \neq 0.$ Hence, $\rho(A) = 2.$

Question : Find the rank of matrix

$$(i)\begin{bmatrix} 2 & 4 & 3 & 2 \\ 3 & 5 & 1 & 4 \end{bmatrix}_{2\times 4} \qquad (ii)\begin{bmatrix} 2 & 1 & -1 \\ 0 & 3 & -2 \\ 2 & 4 & -3 \end{bmatrix}_{3\times 3}$$

$$(iii)\begin{bmatrix} 1 & 3 & 2 \\ 1 & 2 & 3 \\ 1 & 5 & 4 \end{bmatrix}_{3\times 3} \qquad (iv)\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}_{3\times 3}$$

Example: Find the rank of matrix $A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 4 & 6 & 2 \\ 1 & 2 & 3 & 2 \end{bmatrix}_{3\times 4}$

Solution: Since, A is of order 3 x 4.

Then rank of A $\leq$ min *(3, 4)*

rank A $\leq$ 3

Now, the minors of order 3 are the following

$$\begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 2 & 3 \end{vmatrix}, \quad \begin{vmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 2 \end{vmatrix}, \quad \begin{vmatrix} 1 & 3 & 1 \\ 2 & 6 & 2 \\ 1 & 3 & 2 \end{vmatrix}, \quad \begin{vmatrix} 2 & 3 & 1 \\ 4 & 6 & 2 \\ 2 & 3 & 2 \end{vmatrix}$$

Clearly, all minors of order 3 are zero.

Again, consider minor of order 2

$$\begin{vmatrix} 6 & 2 \\ 3 & 2 \end{vmatrix} = 6 \neq 0.$$

Hence, $\rho(A) = 2.$

Question : Find the rank of matrix

$$(i)\begin{bmatrix}1 & 2 & 3 & 2\\ 2 & 3 & 5 & 1\\ 1 & 3 & 4 & 5\end{bmatrix}_{3\times 4} \qquad (ii)\begin{bmatrix}1 & 2 & 0\\ 3 & 7 & 1\\ 5 & 9 & 3\end{bmatrix}_{3\times 3}$$

$$(iii)\begin{bmatrix}1 & 2 & 3\\ 1 & 4 & 2\\ 2 & 6 & 5\end{bmatrix}_{3\times 3} \qquad (iv)\begin{bmatrix}1 & 1 & 1\\ 1 & 2 & 3\\ 1 & 4 & 9\end{bmatrix}_{3\times 3}$$

Example: If the rank of $\begin{bmatrix}2 & 1 & -1\\ 1 & 4 & 2\\ 3 & 5 & k\end{bmatrix}_{3\times 3}$ is 2, find the value of k.

Solution: Let $A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 4 & 2 \\ 3 & 5 & k \end{bmatrix}_{3\times 3}$

Since, rank $A = 2$ then $|A| = 0$

$$\begin{vmatrix} 2 & 1 & -1 \\ 1 & 4 & 2 \\ 3 & 5 & k \end{vmatrix} = 0$$

$$2(4k - 10) - 1(k - 6) - 1(5 - 12) = 0$$

$$7k = 7$$

$$k = 1$$

Methods to find the rank of a Matrix

There are three methods to find the rank of matrix:

1. Minor Method (Definition of rank)

2. Echelon Method (Applying only row operations)

3. Normal Method or Canonical form (Applying rows as well as columns operation)

Echelon form of a Matrix

A matrix A is called in Echelon form if

(i) Every row of A which has all its entries 0 occurs below every row which has a non- zero entry.

(ii) The first non-zero entry in each non-zero row is equal to 1.

(iii) The number of zeros before the first non-zero element in a row is less than the number of such zeros in the next row.

In other words

$$A = \begin{bmatrix} a_{11} = 1 & a_{12} & a_{13} \\ a_{21} = 0 & a_{22} = 1 & a_{23} \\ a_{31} = 0 & a_{32} = 0 & a_{33} = 1 \end{bmatrix}_{3\times 3}$$

Example: $A = \begin{bmatrix} 1 & 0 & 2 & 0 & 7 \\ 0 & 1 & 5 & 0 & 5 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{4\times 5} \Rightarrow \rho(A) = 3$

Important Result: The rank of the matrix in Echelon form is equal to the number of non-zero rows of the matrix.

Example: Find the rank of matrix

$$A = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 8 \\ 0 & 0 & 0 & 0 \end{bmatrix}_{3\times 4}$$

Solution: The matrix A is in Echelon form

So, rank A = the number of non-zeros rows of A

rank A = 2

Example 1: Determine the rank of the matrix by Echelon form

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}_{3\times 3}$$

Solution: The given matrix is $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}_{3\times 3}$

Applying row operations

$$R_2 \to R_2 - R_1, \;\; R_3 \to R_3 - 2R_1$$

$$A \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -1 \\ 0 & 2 & -1 \end{bmatrix}_{3\times 3}$$

$$R_3 \to R_3 - R_2, \;\; R_1 \to R_1 - R_2$$

$$A \sim \begin{bmatrix} 1 & 0 & 4 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix}_{3\times 3}$$

$$R_2 \to \frac{R_2}{2}$$

$$A \sim \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}_{3\times 3}$$

This is required Echelon form and it has two non-zeros rows

Hence, rank $A = 2$.

Example 2: Find the rank of the matrix by Echelon form

$$A = \begin{bmatrix} 1 & 2 & -1 & 4 \\ 2 & 4 & 3 & 5 \\ -1 & -2 & 6 & 7 \end{bmatrix}_{3\times 4}$$

Solution: Given, $A = \begin{bmatrix} 1 & 2 & -1 & 4 \\ 2 & 4 & 3 & 5 \\ -1 & -2 & 6 & 7 \end{bmatrix}_{3\times 4}$

Applying row operations

$$R_2 \to R_2 - 2R_1, \;\; R_3 \to R_3 + R_1$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 0 & 5 & -3 \\ 0 & 0 & 5 & 11 \end{bmatrix}_{3\times 4}$$

$$R_2 \to \frac{R_2}{5}$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 0 & 1 & -\frac{3}{5} \\ 0 & 0 & 5 & 11 \end{bmatrix}_{3\times 4}$$

$$R_3 \rightarrow R_3 - 5R_2$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 0 & 1 & -\frac{3}{5} \\ 0 & 0 & 0 & 14 \end{bmatrix}_{3\times 4}$$

$$R_3 \rightarrow \frac{R_3}{14}$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 0 & 1 & -\frac{3}{5} \\ 0 & 0 & 0 & 1 \end{bmatrix}_{3\times 4}$$

This is required Echelon form and it has three non-zeros rows

Hence, $\text{rank } A = 3$.

Question: Find the rank of a matrix by Echelon form

$$(i)\begin{bmatrix}1 & 3 & 4 & 3\\ 3 & 9 & 12 & 9\\ 1 & 3 & 4 & 1\end{bmatrix}_{3\times 4} \qquad (ii)\begin{bmatrix}3 & -1 & 2\\ -6 & 2 & 4\\ -3 & 1 & 2\end{bmatrix}_{3\times 3}$$

$$(iii)\begin{bmatrix}5 & 3 & 14 & 4\\ 0 & 1 & 2 & 1\\ 1 & -1 & 2 & 0\end{bmatrix}_{3\times 4} \qquad \text{(iv)}\begin{bmatrix}1 & 3 & -1 & 8\\ -4 & 2 & 0 & 0\\ 3 & 1 & 2 & 1\\ 1 & 1 & 1 & 6\end{bmatrix}_{4\times 4}$$

Normal or Canonical form of Matrix

By a finite number of elementary transformation every non-zero matrix of order m x n can be reduced of the following forms:

(i) $\left[\begin{array}{c|c} I_r & O \\ \hline O & O \end{array}\right]$ (ii) $\left[\begin{array}{c} I_r \\ \hline O \end{array}\right]$

(iii) $\left[\begin{array}{c|c} I_r & O \end{array}\right]$ (iv) $[I_r]$

Where I_r is identity matrix of order r. Each one of those four forms is called as Normal form or Canonical form of matrix.

The number r so obtained is called the rank A $i.e.\ \rho(A) = r$

In order to reduce a matrix into its normal form both row and column transformation may be used.

Example 1: If a matrix A can be reduced normal form $\begin{bmatrix} I_5 & O \\ O & O \end{bmatrix}$ by using elementary transformation then find $\rho(A)$.

Solution: Since, $A \sim \begin{bmatrix} I_5 & O \\ O & O \end{bmatrix}$ where I_5 is identity matrix of order 5.

Hence, $\rho(A) = 5.$

Example 2: Let $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}_{4\times 4}$ then show that

(i) $\rho(A) = 3$

(ii) $\rho(A^2) = 2$

(iii) $\rho(A^3) = 1$

(iv) $\rho(A^4) = 0$

Solution: (i) $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}_{4\times4}$

Apply $C_1 \leftrightarrow C_2$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}_{4\times4}$$

Applying $C_2 \leftrightarrow C_3$ and again applying $C_3 \leftrightarrow C_4$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}_{4\times4}$$

$$A \sim \begin{bmatrix} I_3 & O \\ O & O \end{bmatrix}$$

Hence, $\rho(A) = 3.$

$$\text{(ii) } A^2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}_{4\times 4} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}_{4\times 4}$$

$$A^2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}_{4\times 4}$$

Applying $C_1 \leftrightarrow C_2$ and again applying $C_2 \leftrightarrow C_4$

$$A^2 \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}_{4\times 4}$$

$$A^2 \sim \begin{bmatrix} I_2 & O \\ O & O \end{bmatrix}$$

Hence, $\rho(A^2) = 2.$

Similarly, we can find easily $\boldsymbol{\rho(A^3) = 1}$ *and* $\boldsymbol{\rho(A^4) = 0}$.

Example 3. Reduce the following matrix to its normal form and find its rank

$$\begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}_{3\times 3}$$

<u>Solution:</u>

$$A = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}_{3\times 3}$$

$$R_2 \to R_2 - R_1, \;\; R_3 \to R_3 - R_1$$

$$\sim \begin{bmatrix} 2 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{3\times 3}$$

$$C_2 \to C_2 - C_1, \;\; C_3 \to C_3 - C_1$$

$$\sim \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{3\times 3}$$

$$C_2 \to \frac{1}{2} C_1,$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{3\times 3}$$

$$A \sim \begin{bmatrix} I_1 & O \\ O & O \end{bmatrix}$$

Hence, $\rho(A) = 1.$

Example 4. Reduce the following matrix to its normal form and find its rank

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{bmatrix}_{3\times 3}$$

Solution:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{bmatrix}_{3\times 3}$$

$$C_2 \to C_2 - C_1, \quad C_3 \to C_3 - C_2$$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 3 & 2 & 2 \end{bmatrix}_{3\times 3}$$

$$R_2 \to R_2 - R_1, \quad R_3 \to R_3 - R_2$$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}_{3\times 3}$$

$$R_3 \to R_3 - R_1$$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{3\times 3}$$

$$C_3 \to C_3 - C_2$$

$$\sim \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{3\times 3}$$

$$C_1 \to C_1 - C_2$$

$$\sim \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{3\times 3}$$

$$C_1 \leftrightarrow C_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{3\times3}$$

$$A \sim \begin{bmatrix} I_2 & O \\ O & O \end{bmatrix}$$

Hence, $\rho(A) = 2.$

Example 5. Reduce the following matrix to its normal form and find its rank

$$\begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}_{4\times4}$$

Solution: Given $A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}_{4\times4}$

$$C_3 \to C_3 - C_1, \quad C_4 \to C_4 - C_1$$

$$\sim \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 0 & 0 \\ 3 & 1 & -3 & -1 \\ 1 & 1 & -3 & -1 \end{bmatrix}_{4\times 4}$$

$$R_3 \to R_3 - R_1,\;\; R_4 \to R_4 - R_1$$

$$\sim \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}_{4\times 4}$$

$$C_3 \to C_3 + 3C_2,\;\; C_4 \to C_4 + C_2$$

$$\sim \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}_{4\times 4}$$

$$R_3 \to R_3 - 3R_2,\;\; R_4 \to R_4 - R_2$$

$$\sim \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}_{4\times 4}$$

$$C_1 \leftrightarrow C_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}_{4\times 4}$$

$$A \sim \begin{bmatrix} I_2 & O \\ O & O \end{bmatrix}$$

Hence, $\rho(A) = 2.$

Question: Reduce the following matrix into normal form and find its rank

$$(i)\begin{bmatrix}1 & 2 & 3 & 2\\ 2 & 3 & 5 & 1\\ 1 & 3 & 4 & 5\end{bmatrix}_{3\times 4} \qquad (ii)\begin{bmatrix}1 & 2 & 3\\ 2 & 3 & 4\\ 3 & 4 & 5\end{bmatrix}_{3\times 3}$$

$$(iii)\begin{bmatrix}1 & 2 & 3\\ 1 & 4 & 2\\ 2 & 6 & 5\end{bmatrix}_{3\times 3} \qquad (iv)\begin{bmatrix}4 & 2 & 3\\ 8 & 4 & 6\\ -2 & -1 & -\frac{3}{2}\end{bmatrix}_{3\times 3}$$

$$(v)\begin{bmatrix}2 & 3 & -1 & -1\\ 1 & -1 & -2 & -4\\ 3 & 1 & 3 & -2\\ 6 & 3 & 0 & -7\end{bmatrix}_{4\times 4}$$

Solution of Linear Equation

System of Linear Equations

Consider the system of *m* linear equations in *n* unknowns $x_1, x_2, \dots, x_n$

$$\left.\begin{array}{r} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{array}\right\} \dots.(1)$$

Where $a_{ij}(i = 1,2,\dots,m, j = 1,2,\dots,n)$ and $b_i(i = 1,2,\dots,m)$ are constants.

This system of equations is known as a **system of linear equations**.

The above system of equations (1) can be written in matrix form:

$$\begin{bmatrix} a_{11} & a_{12} \dots\dots a_{1n} \\ a_{21} & a_{22} \dots\dots a_{2n} \\ & \vdots \\ a_{m1} & a_{m2} \dots\dots a_{mn} \end{bmatrix}_{m\times n} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_n \end{bmatrix}_{n\times 1} = \begin{bmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ b_m \end{bmatrix}_{m\times 1}$$

Or $A \,.\, X = B$

Where *A* is called the **coefficient Matrix** of system of equation (1)

And *X* is the **column matrix of unknowns variables.**

And *B* is the **column matrix of constants.**

The matrix of coefficient $[a_{ij}]_{m\times n}$ adjoined by the column $\begin{bmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ b_m \end{bmatrix}_{m\times 1}$ is

called **Augmented matrix** and will be denoted by **[*A* : *B*].**

$$i.e. \qquad [A:B] = \begin{bmatrix} a_{11} & a_{12} \ldots\ldots a_{1n} & \vdots & b_1 \\ a_{21} & a_{22} \ldots\ldots a_{2n} & \vdots & b_2 \\ & \vdots & & \\ a_{m1} & a_{m2} \ldots\ldots a_{mn} & \vdots & b_m \end{bmatrix}_{m\times n+1}$$

There are mainly **two types** of system of linear equations.

(i) Non- Homogeneous system of linear equations:

If $B \neq O$ $i.e.$ at least one $b_i \neq 0$ $(i = 1,2, \ldots m)$

$i.e.$ $\quad AX = B(\neq O)$

(ii) Homogeneous system of linear equations:

If $B = O$ (Zero Matrix) $i.e.$ $b_1 = b_2 = \cdots = b_m = 0$

$i.e.$ $\quad AX = O$

Solution of the system of Linear equations:

A set of values of $x_1, x_2, \ldots, x_n$ which satisfy each of above linear equations (1) is called as a solution of the system of linear equations.

Example:
$$\left.\begin{aligned} x + y &= 4 \\ x - y &= 2 \end{aligned}\right\} \ldots \ldots (P)$$

Solve, we get $x = 3, y = 1.$

Here $x = 3, y = 1$ is solution of system (P).

Consistent and Inconsistent System

Consistent: A system of linear equations is called as consistent if there exists at least one solution of the system.

i.e. If the system of equation having one or more solution is called a consistent system of equation.

(i) **Unique Solution:** If the system has only one solution , then we say that the system has unique solution.

Example:

$$\left.\begin{array}{l} x + y = 5 \\ x - y = 1 \end{array}\right\} \ldots\ldots (2)$$

Solve, then we get $x = 3, y = 2$.

The system (2) has unique solution.

(ii) **Infinite Many solution:** If the system has more than one solution then we say that the system has infinite many solution.

Example:

$$\left.\begin{aligned} 2x + 5y &= 3 \\ 4x + 10y &= 6 \end{aligned}\right\} \ldots\ldots.(3)$$

Solve we get (1, 1/5), (2, -1/5),………

Hence, The system (3) have infinite many solution.

- **Both the system (2) and (3) is consistent.**

Inconsistent: If the system of equation having **no solution** is called an inconsistent system of equations.

Example:

$$\left.\begin{aligned} 2x + 5y &= 3 \\ 4x + 10y &= 2 \end{aligned}\right\} \ldots\ldots.(4)$$

In the system (4) has no solution. Hence, the system (4) is inconsistent.

Solution of Non-Homogeneous Linear equations

$A\ X = B$ on the basis of rank

(a) Consistency of System:

If $\rho([A:B]) = \rho(A)$

then the system is consistent.

(i) Unique Solution:

If $\rho([A:B]) = \rho(A) = n$ (No. of unknown variables)

then the system has unique solution.

(ii) Infinite many solution:

If $\rho([A:B]) = \rho(A) < n$ (No. of unknown variables)

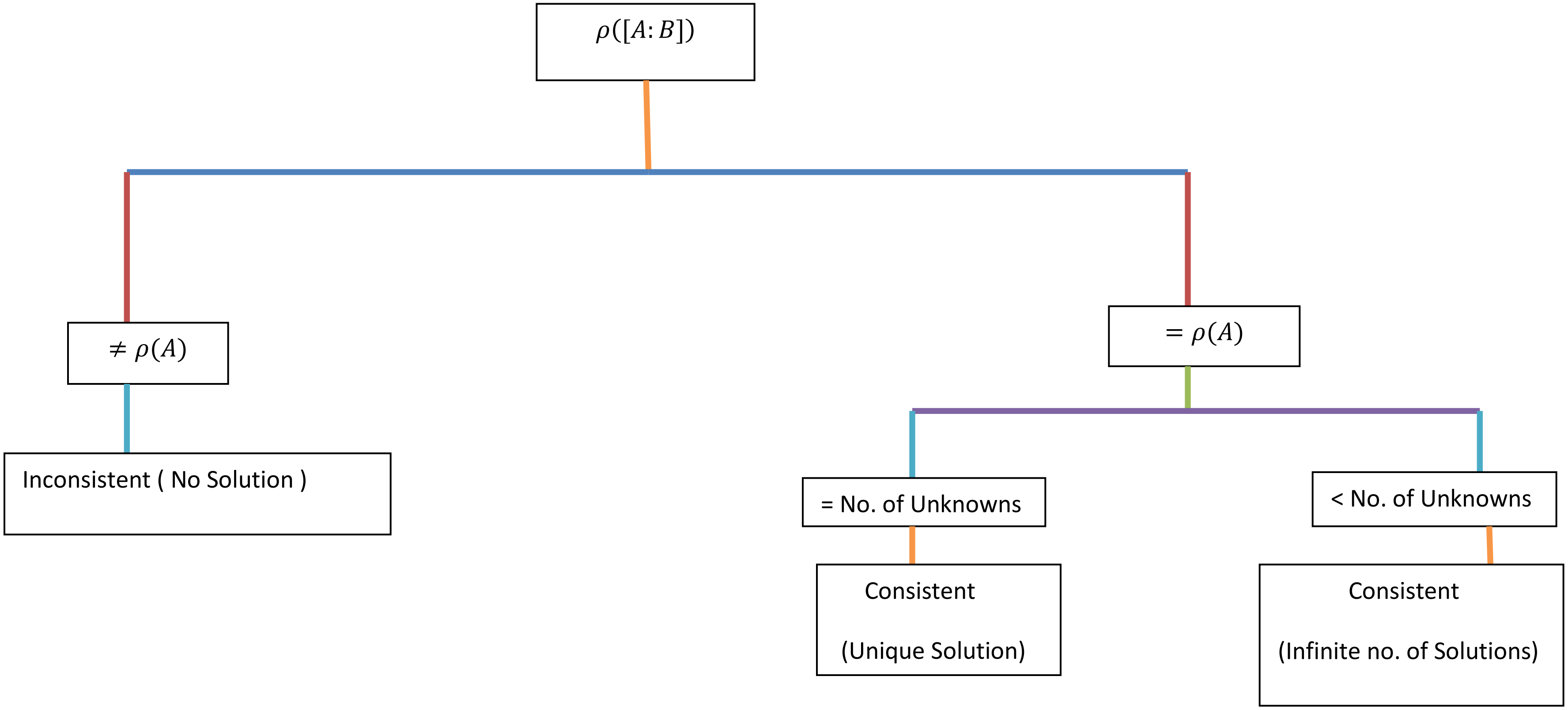
then the system has infinite many solution.

(b) Inconsistent of system:

$$\text{If } \rho([A:B]) \neq \rho(A)$$

then the system is an inconsistent and has no solution.

❖ All the above cases for the system of non-homogeneous linear equations can be collected below in the form of diagram.



Example 1: Show that the system of equation $\left.\begin{matrix} x+y=2 \\ 4x+3y=7 \\ x-2y=1 \end{matrix}\right\}$ is inconsistent.

Solution: Consider the coefficient matrix $A = \begin{bmatrix} 1 & 1 \\ 4 & 3 \\ 1 & -2 \end{bmatrix}$ and

augmented matrix $[A:B] = \left[\begin{array}{cc:c} 1 & 1 & 2 \\ 4 & 3 & 7 \\ 1 & -2 & 1 \end{array}\right]$

Determinant of augmented matrix $= \begin{vmatrix} 1 & 1 & 2 \\ 4 & 3 & 7 \\ 1 & -2 & 1 \end{vmatrix} = -2 \neq 0$

Hence, $\rho([A:B]) = 3$

Since, A is matrix of order 3 x 2.

So, $\rho(A) \leq \min(3,2)$

Hence, $\rho(A) \leq 2$

Hence, rank of coefficient matrix cannot be 3.

Therefore, $\rho([A:B]) \neq \rho(A)$

Given, system of equation is inconsistent. It has no solution.

Example 2: Test for the consistency of $\left.\begin{aligned} x-y+2z&=2\\ 2x+y+4z&=7\\ 4x-y+z&=4 \end{aligned}\right\}$.

Solution: Consider the coefficient matrix $A=\begin{bmatrix}1 & -1 & 2\\ 2 & 1 & 4\\ 4 & -1 & 1\end{bmatrix}$ and

augmented matrix $[A:B]=\left[\begin{array}{ccc:c}1 & -1 & 2 & 2\\ 2 & 1 & 4 & 7\\ 4 & -1 & 1 & 4\end{array}\right]$

Applying $R_2\to R_2-2R_1, R_3\to R_3-4R_1$

$$\sim\left[\begin{array}{ccc:c}1 & -1 & 2 & 2\\ 0 & 3 & 0 & 3\\ 0 & 3 & 7 & -4\end{array}\right]$$

$R_3\to R_3-R_2$

$$\sim\left[\begin{array}{ccc:c}1 & -1 & 2 & 2\\ 0 & 3 & 0 & 3\\ 0 & 0 & -7 & -7\end{array}\right]$$

$R_2\to \frac{1}{3}R_2, R_3\to \frac{-1}{7}R_3$

$$\sim \begin{bmatrix} 1 & -1 & 2 & \vdots & 2 \\ 0 & 1 & 0 & \vdots & 1 \\ 0 & 0 & 1 & \vdots & 1 \end{bmatrix}$$

Hence, $\rho([A:B]) = \rho(A) = 3 =$ No. of unknowns

Then the system of equation is consistent and has unique solution.

Example 3: Show that the system of equation $\left.\begin{aligned} x + 2y + 3z &= 4 \\ 2x + 3y + 8z &= 7 \\ x - y + 9z &= 1 \end{aligned}\right\}$ is consistent and have infinite solutions.

Solution: Consider the coefficient matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 8 \\ 1 & -1 & 9 \end{bmatrix}$ and

augmented matrix $[A:B] = \begin{bmatrix} 1 & 2 & 3 & \vdots & 4 \\ 2 & 3 & 8 & \vdots & 7 \\ 1 & -1 & 9 & \vdots & 1 \end{bmatrix}$

Applying $R_2 \to R_2 - 2R_1, R_3 \to R_3 - R_1$

$$\sim \begin{bmatrix} 1 & 2 & 3 & \vdots & 4 \\ 0 & -1 & 2 & \vdots & -1 \\ 0 & -3 & 6 & \vdots & -3 \end{bmatrix}$$

$R_2 \to (-1)R_2$

$$\sim \begin{bmatrix} 1 & 2 & 3 & \vdots & 4 \\ 0 & 1 & -2 & \vdots & 1 \\ 0 & -3 & 6 & \vdots & 3 \end{bmatrix}$$

$$R_3 \to R_3 + 3R_2$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & \vdots & 4 \\ 0 & 1 & -2 & \vdots & 1 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}$$

The last matrix is in Echelon form in which no. of non-zeros row is 2, both for the matrix A and Augmented matrix $[A{:}B]$

Hence, $\rho([A:B]) = \rho(A) = 2 <$ No. of unknowns

Then the system of equation is consistent and has infinite solutions.

Example 4: Find the value of k of which the following system of equations

will be consistent $$\left.\begin{aligned} 2x + y + 3z &= 1 \\ x - y + 2z &= -3 \\ 4x - y + 7z &= k \end{aligned}\right\}.$$

Solution: Consider the coefficient matrix $A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 2 \\ 4 & -1 & 7 \end{bmatrix}$ and augmented matrix $[A: B] = \left[\begin{array}{ccc:c} 2 & 1 & 3 & 1 \\ 1 & -1 & 2 & -3 \\ 4 & -1 & 7 & k \end{array}\right]$

Applying $R_2 \to 2R_2 - R_1, R_3 \to R_3 - 2R_1$

$$\sim \left[\begin{array}{ccc:c} 2 & 1 & 3 & 1 \\ 0 & -3 & 1 & -7 \\ 0 & -3 & 1 & k-2 \end{array}\right]$$

$R_3 \to R_3 - R_2$

$$\sim \left[\begin{array}{ccc:c} 2 & 1 & 3 & 1 \\ 0 & -3 & 1 & -7 \\ 0 & 0 & 0 & k+5 \end{array}\right]$$

If $k + 5 = 0$; all the element of the lastrow becomes zeros and

Hence, $\rho([A: B]) = \rho(A) = 2 <$ No. of unknowns

Therefore, if $k = -5$, then the system is consistent and has infinte no. of solutions.

- In above example if $k \neq -5$ then $\rho([A: B]) = 3,\ \rho(A) = 2$

$$\rho([A: B]) \neq \rho(A)$$

System of equation is inconsistent *i.e.* it has no solution in this case.

Example 5: Solve the equation with the help of matrices

$$\left.\begin{aligned} x + y + z &= 3 \\ x + 2y + 3z &= 4 \\ x + 4y + 9z &= 6 \end{aligned}\right\}.$$

Solution: The given system of equation can be written in matrix form

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix}$$

Or $$A\,X = B$$

The augmented matrix $[A:B] = \left[\begin{array}{ccc:c} 1 & 1 & 1 & 3 \\ 1 & 2 & 3 & 4 \\ 1 & 4 & 9 & 6 \end{array}\right]$

Applying $R_2 \to R_2 - R_1, R_3 \to R_3 - R_1$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 3 & 8 & 3 \end{array}\right]$$

$$R_3 \to R_3 - 3R_2$$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 2 & 0 \end{array}\right]$$

$$R_1 \to R_1 - R_2, R_3 \to \frac{1}{2} R_3$$

$$\sim \left[\begin{array}{ccc:c} 1 & 0 & -1 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$R_1 \to R_1 + R_3, R_2 \to R_2 - 2R_3$$

$$\sim \left[\begin{array}{ccc:c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

The last matrix is in reduced Echelon form in which no. of non-zeros row is 3, both for the matrix A and Augmented matrix $[A:B]$

Hence, $\rho([A:B]) = \rho(A) = 3 =$ No. of unknowns

Then the system of equation is consistent and has unique solution.

The system is reduced form is given by

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

Hence, $x = 2, y = 1, z = 0$ is required unique solution.

Question 1: Solve the equation with the help of matrices

$$\left.\begin{array}{r} x + y + z = 6 \\ x - y + z = 2 \\ 2x + y - z = 1 \end{array}\right\}.$$

Example 6: Solve the equation with the help of matrices

$$\left.\begin{array}{r} x+y+z=6 \\ x+2y+3z=14 \\ x+4y+7z=30 \end{array}\right\}.$$

Solution: The given system of equation can be written in matrix form

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 14 \\ 30 \end{bmatrix}$$

Or $$A\,X = B$$

The augmented matrix $[A:B] = \left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 14 \\ 1 & 4 & 7 & 30 \end{array}\right]$

Applying $R_2 \to R_2 - R_1, R_3 \to R_3 - R_1$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 0 & 3 & 6 & 24 \end{array}\right]$$

$$R_3 \rightarrow R_3 - 3R_2$$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$R_1 \rightarrow R_1 - R_2$$

$$\sim \left[\begin{array}{ccc:c} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The last matrix is in Echelon form in which no. of non-zeros row is 2, both for the matrix A and Augmented matrix $[A{:}B]$

Hence, $\rho([A:B]) = \rho(A) = 2 <$ No. of unknowns

Then the system of equation is consistent and have infinite no. of solutions.

The system is reduced form is given by

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 8 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x - z \\ y + 2z \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 8 \\ 0 \end{bmatrix}$$

$$x - z = 2, y + 2z = 8$$

Then $x = z - 2 \ and \ y = 8 - 2z$

Taking $z = k$

Then $x = k - 2, y = 8 - 2k, z = k$ is general solution of given equation.

Question 2: Solve the equation with the help of matrices

$$\left.\begin{aligned} x - y + 2z &= 2 \\ 3x - y + 2z &= -6 \\ 3x + y + z &= -18 \end{aligned}\right\}.$$

Example 7: Find the value of α and β for which have

$$\left.\begin{aligned} x + y + 2z &= 3 \\ 2x - y + 3z &= 4 \\ 5x - y + \alpha z &= \beta \end{aligned}\right\}$$

(i) An Unique Solution (ii) Infinite many solutions (iii) No solution

Solution: The given system of equation can be written in matrix form

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & -1 & 3 \\ 5 & -1 & \alpha \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ \beta \end{bmatrix}$$

Or $$A\,X = B$$

The augmented matrix $[A{:}B] = \left[\begin{array}{ccc|c} 1 & 1 & 2 & 3 \\ 2 & -1 & 3 & 4 \\ 5 & -1 & \alpha & \beta \end{array}\right]$

Applying $R_2 \to R_2 - 2R_1, R_3 \to R_3 - 5R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 3 \\ 0 & -3 & -1 & -2 \\ 0 & -6 & \alpha - 10 & \beta - 15 \end{array}\right]$$

$R_3 \to R_3 - 2R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 3 \\ 0 & -3 & -1 & -2 \\ 0 & 0 & \alpha - 8 & \beta - 11 \end{array}\right]$$

$$R_2 \to -\frac{1}{3}\,R_2$$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 2 & 3 \\ 0 & 1 & \frac{1}{3} & \frac{2}{3} \\ 0 & 0 & \alpha - 8 & \beta - 11 \end{array}\right]$$

The last matrix is in Echelon form.

Case (I): If $\alpha = 8,\ \beta \neq 11$ then $\rho([A\!:B]) = 3\ and\ \rho(A) = 2$

So, $\rho([A\!:B]) \neq \rho(A)$

Hence, system is inconsistent and it has no solution.

Case (II): If $\alpha = 8,\ \beta = 11$ then $\rho([A\!:B]) = \rho(A) = 2 <$ Number of unknowns

Hence, system is consistent and it have infinie number of solution.

Case (II): If $\alpha \neq 8$ whatever the value of β then

$\rho([A\!:B]) = \rho(A) = 3 =$ Number of unknowns

Hence, system is consistent and it has an unique solution.

Question 3: Find the value of α and β for which have

$$\left.\begin{aligned} x + y + z &= 6 \\ x + 2y + 3z &= 10 \\ x + 2y + \alpha z &= \beta \end{aligned}\right\}$$

(i) An Unique Solution (ii) Infinite many solutions (iii) No solution

Question 4: Find the value of α and β for which have

$$\left.\begin{aligned} 2x + 3y + 5z &= 9 \\ 7x + 3y - 2z &= 8 \\ 2x + 3y + \alpha z &= \beta \end{aligned}\right\}$$

(i) An Unique Solution (ii) Infinite many solutions (iii) No solution

Question 5: Solve the equation with the help of matrices

$$\left.\begin{aligned} 2x + 3y + z = 9 \\ x + 2y + 3z = 6 \\ 3x + y + 2z = 8 \end{aligned}\right\}.$$

Question 6: Solve the equation with the help of matrices

$$\left.\begin{aligned} x - 3y - 8z = -10 \\ 3x + y - 4z = 0 \\ 2x + 5y + 6z = 13 \end{aligned}\right\}.$$

Question 7: Solve the equation with the help of matrices

$$\left.\begin{aligned} 2x - 3y + 7z = 5 \\ 3x + y - 3z = 13 \\ 2x + 19y - 47z = 32 \end{aligned}\right\}.$$

System of Homogeneous Linear Equations

Consider the system of m homogeneous linear equations in n unknowns $x_1, x_2, \ldots, x_n$ given below

$$\left.\begin{array}{c} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = 0 \end{array}\right\} \ldots .(1)$$

Where $a_{ij}(i = 1,2,\ldots,m, j = 1,2,\ldots,n)$ are constants.

This system of equations is known as a **system of Homogeneous linear equations**.

The above system of equations (1) can be written in matrix form:

$$\begin{bmatrix} a_{11} & a_{12} \dots\dots a_{1n} \\ a_{21} & a_{22} \dots\dots a_{2n} \\ & \vdots \\ a_{m1} & a_{m2} \dots\dots a_{mn} \end{bmatrix}_{m\times n} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_n \end{bmatrix}_{n\times 1} = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ 0 \end{bmatrix}_{m\times 1}$$

Or $A \,.\, X = O$

Where A is called the **coefficient Matrix** of system of equation (1)

And X is the **column matrix of unknowns variables.**

This system is **always consistent**.

Since, $x_1 = 0, x_2 = 0, \dots, x_n = 0$ satisfy the system.

Trivial solution or Null solution for *AX*= *O* :

If $x_1 = 0, x_2 = 0, \dots, x_n = 0$ is solution of the system of equation (1) and this solution is known as the trivial solution.

Non-Trivial solution or Non-Zero solution for $AX = O$:

Any other solution $x_1 = \alpha_1, x_2 = \alpha_2, \ldots, x_n = \alpha_n$ where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are not all zero is called a Non-trivial solution.

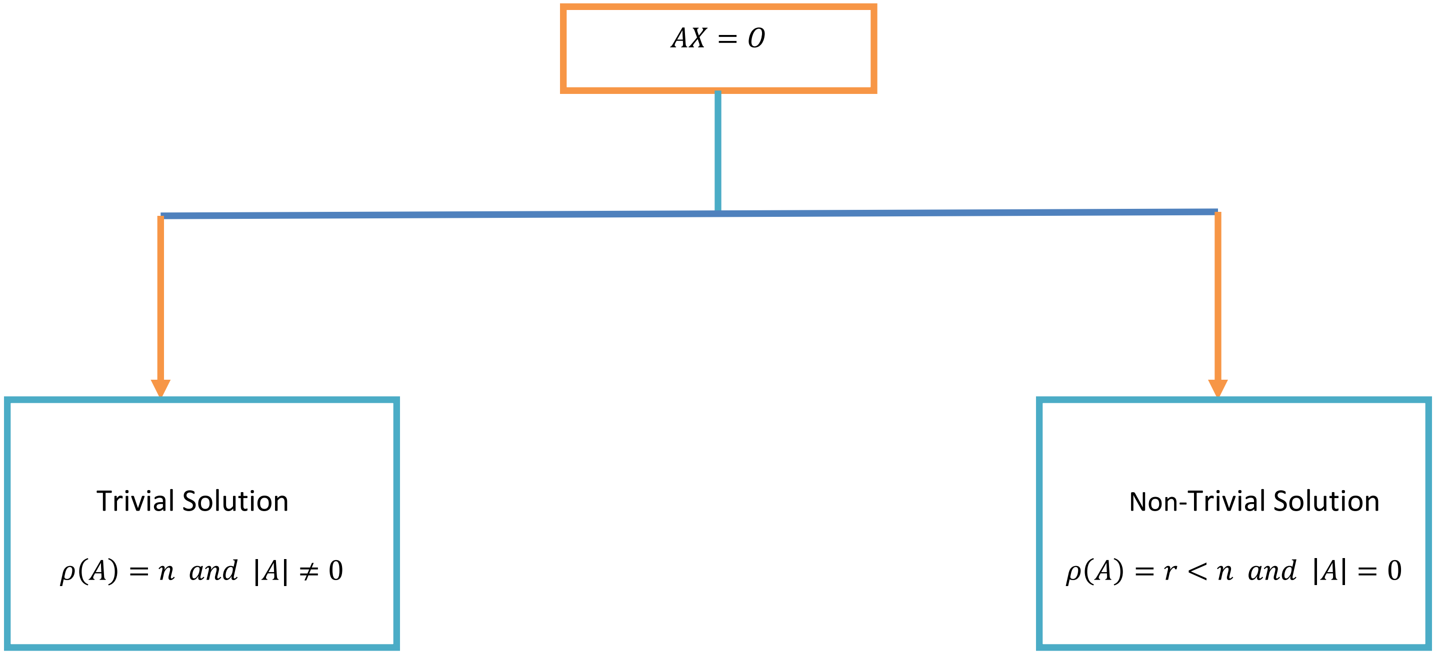
i.e. If the system of equation has infinite many solution then we say that system has non-trivial or non-zero solution.

Nature of solution of the system $A\ X = O$:

A system of homogeneous linear equation has either the trivial solution or non-trivial solution.

(i) If $\rho(A) = n$ (Number of unknowns) then system has only trivial solution.
Hence, $|A| \neq 0$.

(ii) If $\rho(A) = r < n$ (Number of unknowns) then the system has an infinite number of non-trivial solution.
Hence, $|A| = 0$.



$AX = O$
Trivial Solution
$\rho(A) = n \;\; and \;\; |A| \neq 0$
Non-Trivial Solution
$\rho(A) = r < n \;\; and \;\; |A| = 0$

Example 1: Does the following system of equation possesses zero solution

(Trivial solution) ? $$\left.\begin{aligned} 2x - y + z = 0 \\ 3x + 2y + z = 0 \\ x - 3y + 5z = 0 \end{aligned}\right\}.$$

Solution: The given system of equation is equivalent to

$$\begin{bmatrix} 2 & -1 & 1 \\ 3 & 2 & 1 \\ 1 & -3 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$AX = O$$

Consider, $$|A| = \begin{vmatrix} 2 & -1 & 1 \\ 3 & 2 & 1 \\ 1 & -3 & 5 \end{vmatrix} = 29 \neq 0$$

Hence system of equation has only trivial solution

$$x = 0, y = 0, z = 0.$$

Question 1: Find all solution of the following system of equations

(i)
$$\left.\begin{aligned} 2x - 3y + z &= 0 \\ x + 2y - 3z &= 0 \\ 4x - y - 2z &= 0 \end{aligned}\right\}$$

(iii)
$$\left.\begin{aligned} x + 2y + 3z &= 0 \\ 3x + 4y + 4z &= 0 \\ 7x + 10y + 12z &= 0 \end{aligned}\right\}$$

(ii)
$$\left.\begin{aligned} x + 3y - 2z &= 0 \\ 2x - y + 4z &= 0 \\ x - 11y + 14z &= 0 \end{aligned}\right\}$$

(iv)
$$\left.\begin{aligned} x - 2y + z &= 0 \\ x - 2y + z &= 0 \\ 2x - 4y - 5z &= 0 \end{aligned}\right\}$$

Solution of Linear Equation by Cramer's Rule

Cramer's Rule for solving *n* Simultaneous Non-Homogeneous Linear Equations in *n* Unknowns

Consider *n* non-Homogeneous equation in *n* unknowns $x_1, x_2, \dots, x_n$

$$\left.\begin{array}{c} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{n1}x_1 + a_{m2}x_2 + \cdots + a_{nn}x_n = b_n \end{array}\right\} \dots . (1)$$

Where $a_{ij}(i = 1,2, \dots, n, j = 1,2, \dots, n)$ and $b_i(i = 1,2, \dots, n)$ are constants.

The above system of equations (1) can be written in matrix form:

$$\begin{bmatrix} a_{11} & a_{12} \dots\dots a_{1n} \\ a_{21} & a_{22} \dots\dots a_{2n} \\ & \vdots \\ a_{n1} & a_{n2} \dots\dots a_{nn} \end{bmatrix}_{n\times n} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_n \end{bmatrix}_{n\times 1} = \begin{bmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ b_n \end{bmatrix}_{n\times 1}$$

Or $A \,.\, X = B$

If $|A| \neq 0$ *i.e.* A is non- singular.

Then from Cramer's Rule

$$x_1 = \frac{|A_1|}{|A|}, x_2 = \frac{|A_2|}{|A|}, x_3 = \frac{|A_3|}{|A|}, \dots, x_n = \frac{|A_n|}{|A|}$$

Where $|A_1|, |A_2|, \dots, |A_n|$ are determinants obtained by replacing the elements of 1st , 2nd, 3rd, …,*nth* columns of $|A|$ by the element of B.

$$i.e.\quad A_1=\begin{bmatrix} b_1 & a_{12}\ \ldots\ldots a_{1n} \\ b_2 & a_{22}\ \ldots\ldots a_{2n} \\ & \vdots \\ b_n & a_{n2}\ \ldots\ldots a_{nn} \end{bmatrix},\quad A_2=\begin{bmatrix} a_{11} & b_1\ \ldots\ldots a_{1n} \\ a_{21} & b_2\ \ldots\ldots a_{2n} \\ & \vdots \\ a_{n1} & b_n\ \ldots\ldots a_{nn} \end{bmatrix},\ldots$$

$$A_n=\begin{bmatrix} a_{11} & a_{12}\ \ldots\ldots b_1 \\ a_{21} & a_{22}\ \ldots\ldots b_2 \\ & \vdots \\ a_{n1} & a_{n2}\ \ldots\ldots b_n \end{bmatrix}$$

❖ **This method is fail when $|A| = 0$**

Example 1: Can the following equations be solved by Cramer's rule?

Given reason for your answer $\left.\begin{aligned} x+2y+3z&=1\\ 4x+5y+6z&=2\\ 7x+8y+10z&=3 \end{aligned}\right\}.$

Solution: The given system of equation is equivalent to

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Or $$A\,X = B$$

The coefficient matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{bmatrix}$

So, $|A| = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{vmatrix} = 1(50-48) - 2(40-42) + 3(32-35) = -3 \neq 0$

Hence, $|A| \neq 0 \quad i.e.$ A is non-singular

Therefore, given equation can be solved by Cramer's rule.

Example 2: Solve the following equations by Cramer's rule

$$\left.\begin{array}{c} x + y + z = 9 \\ 2x + 5y + 7z = 52 \\ 2x + y - z = 0 \end{array}\right\}.$$

Solution: The given system of equation can be written as

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 5 & 7 \\ 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 52 \\ 0 \end{bmatrix}$$

Or $$A\,X = B$$

The coefficient matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 5 & 7 \\ 2 & 1 & -1 \end{bmatrix}$

So, $|A| = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 5 & 7 \\ 2 & 1 & -1 \end{vmatrix} = 1(-5-7) - 2(-1-1) + 2(7-5) = -4 \neq 0$

Hence, $|A| \neq 0 \quad i.e.$ A is non-singular

Therefore, given equation can be solved by Cramer's rule.

$$|A_1| = \begin{vmatrix} 9 & 1 & 1 \\ 52 & 5 & 7 \\ 0 & 1 & -1 \end{vmatrix} = -4$$

$$|A_2| = \begin{vmatrix} 1 & 9 & 1 \\ 2 & 52 & 7 \\ 2 & 0 & -1 \end{vmatrix} = -12$$

$$|A_3| = \begin{vmatrix} 1 & 1 & 9 \\ 2 & 5 & 52 \\ 2 & 1 & 0 \end{vmatrix} = -20$$

Hence, from Cramer's rule

$$x = \frac{|A_1|}{|A|} = \frac{-4}{-4} = 1$$

$$y = \frac{|A_2|}{|A|} = \frac{-12}{-4} = 3$$

$$z = \frac{|A_3|}{|A|} = \frac{-20}{-4} = 5$$

We get $x = 1, y = 3, z = 5$.

Question : Solve the following equations by Cramer's rule

$$(i)\quad \begin{aligned} x + 2y + 3z &= 4 \\ 2x + 3y + 8z &= 7 \\ x + y + 9z &= 1 \end{aligned} \qquad (ii)\quad \begin{aligned} x + y + z &= 11 \\ 2x - 6y - z &= 0 \\ 3x + 4y + 2z &= 0 \end{aligned} \quad .$$

Eigen Values and Eigen Vectors

Characteristic Value Problem or Eigen Value Problem

Let $A = [a_{ij}]_{n\times n}$ be square matrix of order n .The problem of finding

the scalars and non-zero vectors $X = \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_n \end{bmatrix}_{n\times 1}$

which satisfy the equation $AX = \lambda X$, corresponding to matrix A is known as **Characteristic value problem or Eigen value problem**.

Characteristic Matrix

Let $A = [a_{ij}]_{n\times n}$ be square matrix of order n. The matrix $A - \lambda I$ is called the Characteristic Matrix of A, where λ is scalar and I is the unit matrix of order n.

i.e.

$$A - \lambda I = \begin{bmatrix} a_{11} - \lambda & a_{12} \ldots & \ldots a_{1n} \\ a_{21} & a_{22-\lambda} \ldots \ldots & \ldots a_{2n} \\ & \vdots & \\ a_{n1} & a_{n2} \ldots \ldots & a_{nn} - \lambda \end{bmatrix}_{n\times n}$$

Example:

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

Then Characteristic matrix $A - \lambda I = \begin{bmatrix} 2 - \lambda & 2 & 1 \\ 1 & 3 - \lambda & 1 \\ 1 & 2 & 2 - \lambda \end{bmatrix}$

Characteristic Polynomial

The determinant $|A - \lambda I|$ is called the Characteristic Polynomial of A.

Example: $$\begin{vmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{vmatrix} = \lambda^3 - 7\lambda^2 + 11\lambda - 5$$

This is Characteristic Polynomial of A.

Characteristic Equation

The equation $|A - \lambda I| = 0$ is called the Characteristic Equation of A.

Example: $\lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0$ is Characteristic Equation of matrix A.

Characteristic roots or Eigen values or Latent roots

The roots of Characteristic equation

$$|\boldsymbol{A} - \boldsymbol{\lambda I}| = \mathbf{0}$$

are called Characteristic roots of a matrix A.

Example: $\lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0$

$$(\lambda - 1)(\lambda - 1)(\lambda - 5) = 0$$

$$\lambda = 1, 1, 5$$

are Characteristic roots of given matrix A.

Characteristic Vectors or Eigen Vectors or Latent Vectors

If λ is a characteristic root of a square matrix *A*, then a non-zero vector *X* such that

$$AX = \lambda X$$

is called a characteristic vector or eigen vector or latent vector of *A*, corresponding the root λ.

Example 1: Find the eigen value or characteristic roots of matrix

$$A = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}$$

Solution: Here,

$$A = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}$$

The characteristic matrix

$$A - \lambda I = \begin{bmatrix} 3-\lambda & 4 \\ 4 & -3-\lambda \end{bmatrix}$$

The characteristic equation of matrix A is

$$|A - \lambda I| = \begin{vmatrix} 3-\lambda & 4 \\ 4 & -3-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)(-3-\lambda) - 16 = 0$$

$$\lambda^2 - 25 = 0$$

$$\lambda = 5, -5$$

Here, the eigen values of given matrix A are 5 and -5.

Example 2: Find the eigen value or characteristic roots of matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$

Solution: Here,

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$

and

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 0 & 0 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 3-\lambda \end{bmatrix}$$

The characteristic equation of matrix A is

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 3-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(2-\lambda)(3-\lambda) = 0$$

$$\lambda = 1, 2, 3$$

Here, the eigen values of given matrix A are 1, 2 and 3.

Example 3: Find the eigen values and eigen vectors of the matrix

$$A = \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix}$$

Solution: Here, $A = \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix}$

The characteristic equation of matrix A is

$$|A - \lambda I| = \begin{vmatrix} 3-\lambda & 2 \\ -1 & -\lambda \end{vmatrix} = 0$$

$$(3-\lambda)(-\lambda) + 2 = 0$$

$$\lambda^2 - 3\lambda + 2 = 0$$

$$(\lambda - 1)(\lambda - 2) = 0$$

$$\lambda = 1, 2$$

Here, the eigen values of given matrix A are 1 and 2.

Case- I: Let $X_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ be the eigen vector corresponding to eigen value $\lambda = 1$ of a matrix A then

$$AX_1 = \lambda X_1$$

$$AX_1 - 1.X_1 = O$$

$$(A - 1.I)X_1 = O$$

$$\begin{bmatrix} 3-1 & 2 \\ -1 & 0-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$2x_1 + 2x_2 = 0$$

$$-x_1 - x_2 = 0$$

Then

$$x_1 + x_2 = 0$$

$$x_2 = -x_1$$

$$\frac{x_1}{1} = \frac{x_2}{-1}$$

Then $$X_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Case- II: Let $X_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ be the eigen vector corresponding to eigen value $\lambda = 2$ of a matrix A then

$$AX_2 = \lambda X_2$$

$$AX_2 - 2.X_2 = O$$

$$(A - 2.I)X_2 = O$$

$$\begin{bmatrix} 3-2 & 2 \\ -1 & 0-2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x_1 + 2x_2 = 0$$

$$-x_1 - 2x_2 = 0$$

Then $$x_1 + 2x_2 = 0$$

$$x_1 = -2x_2$$

$$\frac{x_1}{-2} = \frac{x_2}{1}$$

Then $$X_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Hence, eigen vectors are $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ corresponding to eigen value $\lambda = 1, \lambda = 2$ respectively.

Question : Find the eigen values and eigen vectors of the matrix

(i) $A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$ (*ii*) $A = \begin{bmatrix} 1 & -2 \\ -5 & 4 \end{bmatrix}$

Example 4: Find the eigen values and eigen vectors of the matrix

$$A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

Solution: The characteristic equation of matrix *A* is

$$|A - \lambda I| = \begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - 18\lambda^2 + 45\lambda = 0$$

$$\lambda(\lambda - 3)(\lambda - 15) = 0$$

$$\lambda = 0, 3, 15$$

Here, the eigen values of given matrix A are 0, 3 and 15.

Case- I: Let $X_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be the eigen vector corresponding to eigen value $\lambda = 0$ of a matrix *A* then

$$AX_1 = \lambda X_1$$

$$AX_1 - 0.X_1 = O$$

$$(A - 0.I)X_1 = O$$

$$\begin{bmatrix} 8-0 & -6 & 2 \\ -6 & 7-0 & -4 \\ 2 & -4 & 3-0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The coefficient matrix $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

Applying $R_1 \to \frac{1}{2}R_1, R_2 \to R_2 + R_3$

$$\sim \begin{bmatrix} 4 & -3 & 1 \\ -4 & 3 & -1 \\ 2 & -4 & 3 \end{bmatrix}$$

$$R_2 \to R_2 + R_1$$

$$\sim \begin{bmatrix} 4 & -3 & 1 \\ 0 & 0 & 0 \\ 2 & -4 & 3 \end{bmatrix}$$

$$R_2 \leftrightarrow R_3$$

$$\sim \begin{bmatrix} 4 & -3 & 1 \\ 2 & -4 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore, the system of equation is equivalent to the system

$$\begin{bmatrix} 4 & -3 & 1 \\ 2 & -4 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$4x_1 - 3x_2 + x_3 = 0$$

$$2x_1 - 4x_2 + 3x_3 = 0$$

Solving by the method of cross-multiplication

$$\frac{x_1}{-9+4} = \frac{x_2}{2-12} = \frac{x_3}{-16+6}$$

$$\frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{2}$$

Then $X_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ is an eigen vector corresponding to $\lambda = 0$.

Case- II: Let $X_2 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be the eigen vector corresponding to eigen value $\lambda = 3$ of a matrix A then

$$AX_2 = \lambda X_2$$

$$AX_2 - 3.X_2 = O$$

$$(A - 3.I)X_2 = O$$

$$\begin{bmatrix} 8-3 & -6 & 2 \\ -6 & 7-3 & -4 \\ 2 & -4 & 3-3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The coefficient matrix $\begin{bmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix}$

Applying $R_2 \to -\frac{1}{2} R_2$

$$\sim \begin{bmatrix} 5 & -6 & 2 \\ 3 & -2 & 2 \\ 2 & -4 & 0 \end{bmatrix}$$

$$R_1 \to R_1 - R_2$$

$$\sim \begin{bmatrix} 2 & -4 & 0 \\ 3 & -2 & 2 \\ 2 & -4 & 0 \end{bmatrix}$$

$$R_3 \to R_3 - R_1$$

$$\sim \begin{bmatrix} 2 & -4 & 0 \\ 3 & -2 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$R_1 \to \frac{1}{2} R_1$$

$$\sim\begin{bmatrix}1 & -2 & 0\\ 3 & -2 & 2\\ 0 & 0 & 0\end{bmatrix}$$

Therefore, the system of equation is equivalent to the system

$$\begin{bmatrix}1 & -2 & 0\\ 3 & -2 & 2\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$$x_1 - 2x_2 + 0.x_3 = 0$$

$$3x_1 - 2x_2 + 2x_3 = 0$$

Solving by the method of cross-multiplication

$$\frac{x_1}{2}=\frac{x_2}{1}=\frac{x_3}{-2}$$

Then $X_1 = \begin{bmatrix}2\\ 1\\ -2\end{bmatrix}$ is an eigen vector corresponding to $\lambda = 3$.

Case- III: Let $X_3 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be the eigen vector corresponding to eigen value $\lambda = 15$ of a matrix A then

$$AX_3 = \lambda X_3$$

$$AX_2 - 15.X_3 = O$$

$$(A - 15.I)X_2 = O$$

$$\begin{bmatrix} 8-15 & -6 & 2 \\ -6 & 7-15 & -4 \\ 2 & -4 & 3-15 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The coefficient matrix $\begin{bmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix}$

Applying $R_1 \to -R_1, R_2 \to -\frac{1}{2}R_2, R_3 \to \frac{1}{2}R_3$

$$\sim \begin{bmatrix} 7 & 6 & -2 \\ 3 & 4 & 2 \\ 1 & -2 & -6 \end{bmatrix}$$

$$R_1 \rightarrow R_1 - 2R_2$$

$$\sim \begin{bmatrix} 1 & -2 & -6 \\ 3 & 4 & 2 \\ 1 & -2 & -6 \end{bmatrix}$$

$$R_3 \rightarrow R_3 - R_1$$

$$\sim \begin{bmatrix} 1 & -2 & -6 \\ 3 & 4 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore, the system of equation is equivalent to the system

$$\begin{bmatrix} 1 & -2 & -6 \\ 3 & 4 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 - 2x_2 - 6x_3 = 0$$

$$3x_1 + 4x_2 + 2x_3 = 0$$

Solving by the method of cross-multiplication

$$\frac{x_1}{2} = \frac{x_2}{-2} = \frac{x_3}{1}$$

Then $X_1 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$ is an eigen vector corresponding to $\lambda = 15$.

Hence, eigen vectors are $\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$ corresponding to eigen value $\lambda = 0, \lambda = 3, \lambda = 15$ respectively.

Example 5: Find the eigen values and eigen vectors of the matrix

$$A = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix}$$

Solution: The characteristic equation of matrix A is

$$|A - \lambda I| = \begin{vmatrix} -\lambda & 1 & -1 \\ 1 & -\lambda & -1 \\ 1 & -1 & -\lambda \end{vmatrix} = 0$$

$$\lambda^3 - 3\lambda^2 + 2 = 0$$

$$(\lambda - 1)(\lambda - 1)(\lambda + 2) = 0$$

$$\lambda = 1, 1, -2$$

Here, the eigen values of given matrix A are 1, 1 and -2.

Case- I: Let $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be the eigen vector corresponding to eigen value $\lambda = 1$ of a matrix A then

$$AX = \lambda X$$

$$AX - 1.X = O$$

$$(A - 1.I)X = O$$

$$\begin{bmatrix} -1 & 1 & -1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The coefficient matrix $\begin{bmatrix} -1 & 1 & -1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{bmatrix}$

Applying $R_2 \to R_2 + R_1, R_3 \to R_3 + R_1$

$$\sim \begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$R_1 \to -R_1$

$$\sim \begin{bmatrix} 1 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore, the system of equation is equivalent to the system

$$\begin{bmatrix} 1 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 - x_2 - x_3 = 0$$

then $$x_1 = x_2 + x_3$$

Then $$X_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1.x_2 + 1.x_3 \\ 1.x_2 + 0.x_3 \\ 0.x_2 + 1.x_3 \end{bmatrix}$$

$$= x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$= x_2 X_1 + x_3 X_2$$

Hence, $X_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $X_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ are eigen vector corresponding to $\lambda = 1$.

Case- II: Let $X_3 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be the eigen vector corresponding to eigen value $\lambda = -2$ of a matrix A then

$$AX_3 = \lambda X_3$$

$$AX_3 + 2.X_3 = O$$

$$(A + 2.I)X_3 = O$$

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The coefficient matrix $\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

Applying $R_2 \leftrightarrow R_2$

$$\sim \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 1 \\ 1 & -1 & 2 \end{bmatrix}$$

$R_2 \rightarrow R_2 - 2R_1, R_3 \rightarrow R_3 - R_1$

$$\sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & -3 & 3 \\ 0 & -3 & 3 \end{bmatrix}$$

$R_3 \rightarrow R_3 - R_2$

$$\sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & -3 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$R_2 \to -\frac{1}{2}R_2$$

$$\sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore, the system of equation is equivalent to the system

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 + 2x_2 - x_3 = 0$$

$$x_2 - x_3 = 0$$

Solving by the method of cross-multiplication

$$\frac{x_1}{-1} = \frac{x_2}{1} = \frac{x_3}{1}$$

Then $X_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$ is an eigen vector corresponding to $\lambda = 3$.

Hence, eigen vectors are $\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}-1\\1\\1\end{bmatrix}$ corresponding to eigen value $\lambda = 1, 1, -2$ respectively.

Question : Find the eigen values and eigen vectors of the matrix

$$(i)\ \begin{bmatrix}1 & 1 & 3\\1 & 5 & 1\\3 & 1 & 1\end{bmatrix} \qquad (ii)\ \begin{bmatrix}1 & 0 & 0\\0 & 3 & -1\\3 & -1 & 3\end{bmatrix}$$

Diagonalization

Similarity of Matrices

Let *A* and *B* be square matrices of order *n*. Then *B* is said to be similar to *A* if there exist a non-singular matrix *P* such that

$$\boldsymbol{B = P^{-1}AP}$$

Theorem 1: Similarity of matrices is an equivalence relation.

Proof: If *A* and *B* are two *n* x *n* matrices, then *B* is said to be similar to *A* if there exist a non-singular matrix *P* such that

$$\boldsymbol{B = P^{-1}AP}$$

Reflexivity: We are to prove that every matrix is similar to itself.

Let *A* be any matrix of order *n*. We can write

$$\boldsymbol{A = I^{-1}AI}$$

Where *I* is unit matrix of order *n*.

Therefore A is similar to A.

Symmetry: Let *A* be similar to *B*. Then to prove that *B* is also similar to *A*. Since, *A* is similar to *B*, therefore there exist a non-singular matrix *P* such that $A = P^{-1}BP$

$$PAP^{-1} = P(P^{-1}BP)P^{-1}$$

$$PAP^{-1} = B$$

$$B = PAP^{-1}$$

$$B = (P^{-1})^{-1}AP^{-1}$$

Hence, *B* is similar to *A*.

Transitivity: Let *A* be similar to *B* and *B* be similar to *C*. Then to prove that *A* is similar to *C*. Since, *A* is similar to *B* and *B* be similar to *C* therefore $A = P^{-1}BP$ and $B = Q^{-1}CQ$

Where *P* and *Q* are invertible *n* x *n* matrices. We have

$$A = P^{-1}BP = P^{-1}(Q^{-1}CQ)P$$

$$A = (P^{-1}Q^{-1})C(QP)$$

$$A = (QP)^{-1}C(QP)$$

Hence, *A* is similar to *C*.

Hence, Similarity of matrices is an equivalence relation.

Theorem 2: Similar matrices have the same determinant.

Proof: Suppose, *A* and *B* are similar matrices, therefore there exist a non-singular matrix *P* such that

$$B = P^{-1}AP$$

$$|B| = |P^{-1}AP| = |P^{-1}||A||P|$$

$$|B| = |P^{-1}||P||A| = |P^{-1}P||A| = |I||A|$$

$$|B| = |A|.$$

Theorem 3: Similar matrices have the characteristic polynomial.

Proof: Suppose, A and B are similar matrices, therefore there exist a non-singular matrix P such that

$$B = P^{-1}AP$$

Now consider, $|B - \lambda I| = |P^{-1}AP - \lambda I|$

$$|B - \lambda I| = |P^{-1}AP - P^{-1}\lambda IP| \quad \{\, since, P^{-1}(\lambda I)P = \lambda P^{-1}P = \lambda I\}$$

$$|B - \lambda I| = |P^{-1}(A - \lambda I)P|$$

$$|B - \lambda I| = |P^{-1}||A - \lambda I||P|$$

$$|B - \lambda I| = |P^{-1}P||A - \lambda I|$$

$$|B - \lambda I| = |I||A - \lambda I|$$

$$|B - \lambda I| = |A - \lambda I|$$

It follows that A and b have same characteristic polynomial and hence same eigen value.

Diagonalizable matrix

A matrix A is said to be diagonalizable if **it is similar to a diagonal matrix.**

Thus, a matrix A is diagonalizable if there exist an non-singular matrix (invertible matrix) P such that

$$P^{-1}AP = D$$

Where D is a diagonal matrix.

Theorem 3: An n x n matrix is diagonalizable if and only if it possesses n linearly independent eigen vectors.

Proof: Suppose A is diagonalizable. Then A is similar to a diagonal matrix $D = dia.[\lambda_1, \lambda_2, \dots, \lambda_n]$. Therefore there exists an invertible matrix $P = [X_1, X_2, \dots, X_n]$ such that

$$P^{-1}AP = D$$

$$AP = PD$$

$$A[X_1, X_2, \dots, X_n] \; = [X_1, X_2, \dots, X_n] \; dia.[\lambda_1, \lambda_2, \dots, \lambda_n]$$

$$[AX_1, AX_2, \dots, AX_n] \; = [\lambda_1 X_1, \lambda_2 X_2, \dots, \lambda_n X_n]$$

$$AX_1 = \lambda_1 X_1, AX_2 = \lambda_2 X_2, \dots, AX_n = \lambda_n X_n$$

Therefore, $X_1, X_2, \dots, X_n$ are eigen vectors of A corresponding to the eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ respectively. Since the matrix P is non-singular, therefore its column vectors $X_1, X_2, \dots, X_n$ are linearly independent. Hence A possesses n linearly independent eigen vectors.

Conversely: Suppose that A possesses n linearly independent eigen vectors $X_1, X_2, \dots, X_n$ and let $\lambda_1, \lambda_2, \dots, \lambda_n$ be the corresponding eigenvalues. Then

$$AX_1 = \lambda_1 X_1, \;\; AX_2 = \lambda_2 X_2, \dots, \; AX_n = \lambda_n X_n$$

Let $P = [X_1, X_2, \dots, X_n]$ and $D = dia.[\lambda_1, \lambda_2, \dots, \lambda_n]$.

Then $AP = A[X_1, X_2, \dots, X_n] = [AX_1, AX_2, \dots, A\,X_n]$

$$= [\lambda_1 X_1, \lambda_2 X_2, \dots, \lambda_n X_n]$$

$$= [X_1, X_2, \dots, X_n]\, dia.[\lambda_1, \lambda_2, \dots, \lambda_n]$$

$$= PD$$

Since, the column vectors $X_1, X_2, \dots, X_n$ of the matrix P are linearly independent therefore, P is invertible and P^{-1} exists.

Therefore,

$$AP = PD$$

$$P^{-1}AP = P^{-1}PD$$

$$P^{-1}AP = D$$

Hence, A is similar to a diagonal matrix D.

A is diagonalizable.

Example 1: Reduce the matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$ into a diagonal matrix.

OR

Digonalize the matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$

Solution: Here, $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$

The characteristic matrix

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{bmatrix}$$

The characteristic equation of matrix A is

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(2-\lambda) - 6 = 0$$

$$\lambda^2 - 3\lambda - 4 = 0$$

$$(\lambda - 4)(\lambda + 1) = 0$$

$$\lambda = 4, -1$$

Here, the eigen values of given matrix A are 4 and -1.

Case- I: Let $X_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ be the eigen vector corresponding to eigen value $\lambda = 4$ of a matrix A then

$$AX_1 = \lambda X_1$$

$$AX_1 - 4.X_1 = O$$

$$(A - 4.I)X_1 = O$$

$$\begin{bmatrix} 1-4 & 2 \\ 3 & 2-4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3 & 2 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-3x_1 + 2x_2 = 0$$

$$3x_1 - 2x_2 = 0$$

Then $$3x_1 - 2x_2 = 0$$

$$3x_1 = 2x_2$$

$$\frac{x_1}{2} = \frac{x_2}{3}$$

Then $$X_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

Case- II: Let $X_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ be the eigen vector corresponding to eigen value $\lambda = -1$ of a matrix A then

$$AX_2 = \lambda X_2$$

$$AX_2 + 1.X_2 = O$$

$$(A + 1.I)X_2 = O$$

$$\begin{bmatrix} 1+1 & 2 \\ 3 & 2+1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$2x_1 + 2x_2 = 0$$

$$3x_1 + 3x_2 = 0$$

Then $$x_1 + x_2 = 0$$

$$x_2 = -x_1$$

$$\frac{x_1}{1} = \frac{x_2}{-1}$$

Then $$X_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Hence, eigen vectors of matrix A are $\begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ corresponding to eigen value $\lambda = 4, \lambda = -1$ respectively.

Let $$P = [X_1, X_2]$$

$$P = \begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix}$$

$$P^{-1} = \begin{bmatrix} \frac{1}{5} & \frac{1}{5} \\ \frac{3}{5} & -\frac{2}{5} \end{bmatrix}$$

Hence, $$P^{-1}AP = \begin{bmatrix} \frac{1}{5} & \frac{1}{5} \\ \frac{3}{5} & -\frac{2}{5} \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix}$$

$$P^{-1}AP = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix} = \text{Diagonal matrix}$$

Matrix A is Diagonalizable.

Example 2: Reduce the matrix $A = \begin{bmatrix} 1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$ into a diagonal matrix.

OR

Digonalize the matrix $A = \begin{bmatrix} 1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$

Solution: The characteristic equation of matrix A is

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 2 & -2 \\ 1 & 2-\lambda & 1 \\ -1 & -1 & -\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(\lambda+1)(\lambda-3) = 0$$

$$\lambda = 1, -1, 3$$

Here, the eigen values of given matrix A are 1, -1 and 3.

Case- I: Let $X_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be the eigen vector corresponding to eigen value $\lambda = 1$ of a matrix A then

$$AX_1 = \lambda X_1$$

$$AX_1 - 1.X_1 = O$$

$$(A - 1.I)X_1 = O$$

$$\begin{bmatrix} 1-1 & 2 & -2 \\ 1 & 2-1 & 1 \\ -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 2 & -2 \\ 1 & 1 & 1 \\ -1 & -1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The coefficient matrix $\begin{bmatrix} 0 & 2 & -2 \\ 1 & 1 & 1 \\ -1 & -1 & -1 \end{bmatrix}$

Applying $R_3 \rightarrow R_3 + R_2$

$$\sim \begin{bmatrix} 0 & 2 & -2 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$R_1 \rightarrow -\frac{1}{2} R_1$$

$$\sim \begin{bmatrix} 0 & 1 & -1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore, the system of equation is equivalent to the system

$$\begin{bmatrix} 0 & 1 & -1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$0.x_1 + x_2 + x_3 = 0$$

$$x_1 + x_2 + x_3 = 0$$

Solving by the method of cross-multiplication

$$\frac{x_1}{2} = \frac{x_2}{-1} = \frac{x_3}{-1}$$

$$\frac{x_1}{-2} = \frac{x_2}{1} = \frac{x_3}{1}$$

Then $X_1 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$ is an eigen vector corresponding to $\lambda = 1$.

Case- II: Let $X_2 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be the eigen vector corresponding to eigen value $\lambda = -1$ of a matrix A then

$$AX_2 = \lambda X_2$$

$$AX_2 + 1.X_2 = O$$

$$(A + 1.I)X_2 = O$$

$$\begin{bmatrix} 1+1 & 2 & -2 \\ 1 & 2+1 & 1 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 2 & -2 \\ 1 & 3 & 1 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The coefficient matrix $\begin{bmatrix} 2 & 2 & -2 \\ 1 & 3 & 1 \\ -1 & -1 & 1 \end{bmatrix}$

Applying $R_1 \to -\frac{1}{2} R_1$

$$\sim \begin{bmatrix} 1 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & -1 & 1 \end{bmatrix}$$

$R_3 \to R_3 + R_1$

$$\sim \begin{bmatrix} 1 & 1 & -1 \\ 1 & 3 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore, the system of equation is equivalent to the system

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & 3 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 + x_2 - x_3 = 0$$

$$x_1 + 3x_2 + x_3 = 0$$

Solving by the method of cross-multiplication

$$\frac{x_1}{4} = \frac{x_2}{-2} = \frac{x_3}{2}$$

$$\frac{x_1}{-2} = \frac{x_2}{1} = \frac{x_3}{-1}$$

Then $X_1 = \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix}$ is an eigen vector corresponding to $\lambda = -1$.

Case- III: Let $X_3 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be the eigen vector corresponding to eigen value $\lambda = 3$ of a matrix A then

$$AX_3 = \lambda X_3$$

$$AX_3 - 3.X_3 = O$$

$$(A - 3.I)X_2 = O$$

$$\begin{bmatrix} 1-3 & 2 & -2 \\ 1 & 2-3 & 1 \\ -1 & -1 & 1-3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 2 & -2 \\ 1 & -1 & 1 \\ -1 & -1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The coefficient matrix $\begin{bmatrix} -2 & 2 & -2 \\ 1 & -1 & 1 \\ -1 & -1 & -3 \end{bmatrix}$

Applying $R_1 \rightarrow \frac{1}{2} R_1$

$$\sim \begin{bmatrix} -1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & -1 & -3 \end{bmatrix}$$

$$R_2 \rightarrow R_2 + R_1$$

$$\sim \begin{bmatrix} -1 & 1 & -1 \\ 0 & 0 & 0 \\ -1 & -1 & -3 \end{bmatrix}$$

Therefore, the system of equation is equivalent to the system

$$\begin{bmatrix} -1 & 1 & -1 \\ 0 & 0 & 0 \\ -1 & -1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-x_1 + x_2 - x_3 = 0$$

$$-x_1 - x_2 - 3x_3 = 0$$

Solving by the method of cross-multiplication

$$\frac{x_1}{-4} = \frac{x_2}{-2} = \frac{x_3}{2}$$

$$\frac{x_1}{2} = \frac{x_2}{1} = \frac{x_3}{-1}$$

Then $X_1 = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$ is an eigen vector corresponding to $\lambda = 3$.

Hence, eigen vectors of matrix A are $\begin{bmatrix}-2\\1\\1\end{bmatrix}, \begin{bmatrix}-2\\1\\-1\end{bmatrix}, \begin{bmatrix}2\\1\\-1\end{bmatrix}$ corresponding to eigen value $\lambda = 1, \lambda = -1, \lambda = 3$ respectively.

Let

$$P = [X_1, X_2, X_3]$$

$$P = \begin{bmatrix}-2 & -2 & 2\\ 1 & 1 & 1\\ 1 & -1 & -1\end{bmatrix}$$

$$P^{-1} = -\frac{1}{8}\begin{bmatrix}0 & -4 & -4\\ 2 & 0 & 4\\ 2 & -4 & 0\end{bmatrix}$$

Hence,

$$P^{-1}AP = -\frac{1}{8}\begin{bmatrix}0 & -4 & -4\\ 2 & 0 & 4\\ 2 & -4 & 0\end{bmatrix}\begin{bmatrix}1 & 2 & -2\\ 1 & 2 & 1\\ -1 & -1 & 0\end{bmatrix}\begin{bmatrix}-2 & -2 & 2\\ 1 & 1 & 1\\ 1 & -1 & -1\end{bmatrix}$$

$$P^{-1}AP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \text{Diagonal matrix}$$

Matrix A is Diagonalizable.

Question: Reduce the following matrix into a diagonal matrix.

$(i) A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

$(ii) \begin{bmatrix} 4 & 2 & -2 \\ -5 & 3 & 2 \\ -2 & 4 & 1 \end{bmatrix}$

Application of Matrices

Application of Matrices

- Graph theory
  - The adjacency matrix of a finite graph is a basic notion of graph theory.
- Linear combinations of quantum states in Physics
  - The first model of quantum mechanics by Heisenberg in 1925 represented the theory's operators by infinite-dimensional matrices acting on quantum states. This is also referred to as matrix mechanics.
- Computer graphics
  - 4×4 transformation rotation matrices are commonly used in computer graphics.
- Solving linear equations
  - Using Row reduction
  - Cramer's Rule (Determinants)
  - Using the inverse matrix
- Cryptography.

Application of Matrices in Cryptography

Cryptography

- Cryptography, is concerned with keeping communications private.
- Cryptography mainly consists of Encryption and Decryption
- Encryption is the transformation of data into some unreadable form.
  - Its purpose is to ensure privacy by keeping the information hidden from anyone for whom it is not intended, even those who can see the encrypted data.
- Decryption is the reverse of encryption
  - It is the transformation of encrypted data back into some intelligible form.
- Encryption and Decryption require the use of some secret information, usually referred to as a key.
- Depending on the encryption mechanism used, the same key might be used for both encryption and decryption, while for other mechanisms, the keys used for encryption and decryption might be different.

Application of matrix to Cryptography

- One type of code, which is extremely difficult to break, makes use of a large matrix to encode a message.

- The receiver of the message decodes it using the inverse of the matrix.

- This first matrix, used by the sender is called the encoding matrix and its inverse is called the decoding matrix, which is used by the receiver.

Message to be sent:

PREPARE TO NEGOTIATE

and the encoding matrix be

$$\begin{bmatrix} -3 & -3 & -4 \\ 0 & 1 & 1 \\ 4 & 3 & 4 \end{bmatrix}$$

We assign a number for each letter of the alphabet.

Such that A is 1, B is 2, and so on. Also, we assign the number 27 to space between two words. Thus the message becomes:

P	*R*	*E*	*P*	*A*	*R*	*E*	*	*T*	*O*	*	*N*	*E*	*G*	*O*	*T*	*I*	*A*	*T*	*E*
16	18	5	16	1	18	5	27	20	15	27	14	5	7	15	20	9	1	20	5

Encoding

- Since we are using a 3 by 3 matrix, we break the enumerated message above into a sequence of 3 by 1 vectors:

$$\begin{bmatrix}16\\18\\5\end{bmatrix}\begin{bmatrix}16\\1\\18\end{bmatrix}\begin{bmatrix}5\\27\\20\end{bmatrix}\begin{bmatrix}15\\27\\14\end{bmatrix}\begin{bmatrix}5\\7\\15\end{bmatrix}\begin{bmatrix}20\\9\\1\end{bmatrix}\begin{bmatrix}20\\5\\27\end{bmatrix}$$

- Note that it was necessary to add a space at the end of the message to complete the last vector.
- We encode the message by multiplying each of the above vectors by the encoding matrix.
- We represent above vectors as columns of a matrix and perform its matrix multiplication with the encoding matrix

$$\begin{bmatrix}-3 & -3 & -4\\0 & 1 & 1\\4 & 3 & 4\end{bmatrix}\begin{bmatrix}16 & 16 & 5 & 15 & 5 & 20 & 20\\18 & 1 & 27 & 27 & 7 & 9 & 5\\5 & 18 & 20 & 14 & 15 & 1 & 27\end{bmatrix}$$

- We get

$$\begin{bmatrix} -122 & -123 & -176 & -182 & -96 & -91 & -183 \\ 23 & 19 & 47 & 41 & 22 & 10 & 32 \\ 138 & 139 & 181 & 197 & 101 & 111 & 203 \end{bmatrix}$$

- The columns of this matrix give the encoded message.

- Encoding is complete

Transmission

The message is transmitted in a linear form

$$-122,\quad 23,\quad 138,\quad -123,\quad 19,\quad 139,\quad -176,\quad 47,\quad 181,$$
$$-182,\quad 41,\quad 197,\quad -96,\quad 22,\quad 101,\quad -91,\quad 10,\quad 111,$$
$$-183\quad 32\quad 203.$$

Decoding

- To decode the message:
  - The receiver writes this string as a sequence of 3 by 1 column matrices and repeats the technique using the inverse of the encoding matrix.
  - The inverse of this encoding matrix is the decoding matrix.

$$\begin{bmatrix} 1 & 0 & 1 \\ 4 & 4 & 3 \\ -4 & -3 & -3 \end{bmatrix}$$

  - To decode the message, perform the matrix multiplication

$$\begin{bmatrix} 1 & 0 & 1 \\ 4 & 4 & 3 \\ -4 & -3 & -3 \end{bmatrix} \begin{bmatrix} -122 & -123 & -176 & -182 & -96 & -91 & -183 \\ 23 & 19 & 47 & 41 & 22 & 10 & 32 \\ 138 & 139 & 181 & 197 & 101 & 111 & 203 \end{bmatrix}$$

  - Matrix obtained is

$$\begin{bmatrix} 16 & 16 & 5 & 15 & 5 & 20 & 20 \\ 18 & 1 & 27 & 27 & 7 & 9 & 5 \\ 5 & 18 & 20 & 14 & 15 & 1 & 27 \end{bmatrix}$$

Decoded Message

- The columns of this matrix, written in linear form, give the original message

16	18	5	16	1	18	5	27	20	15	27	14	5	7	15	20	9	1	20	5
P	*R*	*E*	*P*	*A*	*R*	*E*	*	*T*	*O*	*	*N*	*E*	*G*	*O*	*T*	*I*	*A*	*T*	*E*

Message received:

PREPARE TO NEGOTIATE